CONVERGENCE OF SUBMARTINGALES TO AN INCREASING PROCESS UNDER DISCRETIZATION OF FILTRATIONS

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I. Introduction.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$ be a filtered probability space, where the filtration $(\mathcal{F}_t) = (\mathcal{F}_t^Y)$ is generated by a càdlàg (right continuous and admitting left limits) process $Y = (Y_t,\, t \in [0,T])$. Note that, in general, $(\mathcal{F}_t)$ is not right-continuous. Let

$$\pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = T\}, \qquad n \in \mathbb{N},$$

be a sequence of refining partitions of an interval $[0,T]$ such that $|\pi_n| := \max_i |t_i^n - t_{i-1}^n| \to 0$, $n \to \infty$. Denote $\mathcal{F}_t^n = \sigma(Y_s^n,\, s \le t)$, where

$$Y_t^n := Y_{t_i^n} \qquad \text{for} \quad t \in [t_i^n, t_{i+1}^n), \qquad Y_T^n := Y_{t_{k_n-1}^n}.$$

We suppose that the set of fixed times of discontinuity of Y is included in the union of π_n.

Given an integrable random variable X and a sequence of random variables X^n, $n \in \mathbb{N}$, converging to X in $L^1(\mathbf{P})$, consider the martingale $M = (M_t = \mathbf{E}(X|\mathcal{F}_t),\ t \in [0,T])$, and the sequence of martingales $M^n = (M_t^n = \mathbf{E}(X^n|\mathcal{F}_t^n),\, t \in [0,T])$, $n \in \mathbb{N}$, with respect to the perturbed filtrations $(\mathcal{F}_t^n)_{t\in[0,T]}$, $n \in \mathbb{N}$. Since $\mathcal{F}_t^n \uparrow \mathcal{F}_t$, $n \to \infty$, for each $t \in [0,T]$, we have that $M_t^n \to M_t$ in probability.

In paper [2] it was proved that : 1) In general, the convergence $M^n \to M^+$ for the Skorokhod topology can fail; see the example of Sect. 2 in [2].

2) If Y is a Markov process (not necessarily continuous), then $M^n \to M^+$ in probability for the Skorokhod topology,([2]Theorem 1).

A more general problem is the following : Suppose that X is a $\mathcal{F}$ adapted càdlàg process, and consider X^n the càdlàg version of processes $\mathbf{E}[X_.|\mathcal{F}_.^n]$: i. e. X^n is the $\mathcal{F}^n$-optional projection of X (see [4], VI-43 and VI-47). The same example in [2] shows that in general we have not the convergence of X^n to X in probability for the J_1 topology. This problem was studied in paper [3], under a general assumption of weak convergence of filtrations ([3] Theorems 1, 2 and 3), generalizing the situation here, when process Y is Markov.

We shall prove in this small note that, without any condition on process Y, we get the desired convergence when X is a continuous increasing process.

For shortening notations, the filtrations $(\mathcal{F}_t^n)$, $(\mathcal{F}_t)$ will be denoted $\mathcal{F}^n$, $\mathcal{F}$.

Theorem. *Let X is a $\mathcal{F}$-adapted, continuous, increasing process. We assume that X_T is square integrable, and $X_0 = 0$. Let us denote $X^n = \mathbf{E}[X_.|\mathcal{F}^n_.]$. Then :*

a) X^n is a positive submartingale, with the canonical decomposition $X^n = M^n + A^n$, where M^n is a square integrable $\mathcal{F}^n$-martingale of pure jumps, and A^n is a continuous increasing $\mathcal{F}^n$-adapted process, such that A^n_T is square integrable.

b) We have the convergences for the uniform topology in $\mathbb{D}$: $X^n \xrightarrow{\mathbf{P}} X$, $M^n \xrightarrow{\mathbf{P}} 0$ and $A^n \xrightarrow{\mathbf{P}} X$.

Proof It will be driven in several steps.

1) We have immediately, for every s and t, with $s \leq t$:

$$\mathbf{E}[X^n_t - X^n_s|\mathcal{F}^n_s] = \mathbf{E}[X_t - X_s|\mathcal{F}^n_s] \geq 0$$

hence the property of $\mathcal{F}^n$-submartingale for X^n.

Let us consider now for every n, the jump process

$$M^n = \sum_{i=1}^{k_n}(\mathbf{E}[X_{t^n_i}|\mathcal{F}^n_{t^n_i}] - \mathbf{E}[X_{t^n_i}|\mathcal{F}^n_{i-1}])1_{\{.\geq t^n_i\}}.$$

One can see that M^n is a square integrable martingale of jumps and that $X^n - M^n$ is a continuous process because the times of jumps of X^n belong to the set of elements t^n_i of partition π^n and that X^n and M^n have the same jumps ; finally between 2 successive t^n_i, A^n is increasing, hence the canonical decomposition given in a).

2) Now we show that the sequence (M^n) is tight and that every limit is a law of continuous martingale.

Let us use the Aldous criterion for tightness ; we are given $\delta > 0$, n, and σ , τ stopping times of filtration $(\mathcal{F}^n)$ such that $\sigma \leq \tau \leq \sigma + \delta$, and let $\varepsilon' > 0$. As previously, the elements of partition π^n are denoted t^n_i or t^n_k for $i, k \leq k_n$.

$$\begin{aligned}\mathbf{P}[|M^n_\tau - M^n_\sigma| > \alpha] &\leq \frac{1}{\alpha^2}\mathbf{E}[(M^n_\tau - M^n_\sigma)^2] \\ &\leq \frac{1}{\alpha^2}\mathbf{E}[\sum_{\sigma\leq s\leq\tau}(\Delta M^n_s)^2].\end{aligned}$$

Then

$$\mathbf{E}[\sum_{\sigma\leq s\leq\tau}(\Delta M^n_s)^2] = \mathbf{E}[\sum_{\sigma\leq t^n_i\leq\tau}(\Delta M^n_{t^n_i})^2]$$

$$= \mathbf{E}[\sum_{\sigma\leq t^n_i\leq\tau}(\mathbf{E}[A_{t^n_i}|\mathcal{F}^n_{t^n_i}] - \mathbf{E}[A_{t^n_i}|\mathcal{F}^n_{t^n_{i-1}}])^2]$$

$$\leq \sum_{k=0}^{k_n}\mathbf{E}[1_{[t^n_k, t^n_k+1[}(\sigma)\sum_{\{i>k; t^n_i - t^n_k \leq\delta\}}(\mathbf{E}[A_{t^n_i}|\mathcal{F}^n_{t^n_i}] - \mathbf{E}[A_{t^n_i}|\mathcal{F}^n_{t^n_{i-1}}])^2]$$

$$\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{[t_k^n, t_k^n+1[}(\sigma) \sum_{\{i>k; t_i^n - t_k^n \leq \delta\}} ((\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n])^2 - (\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2)]$$

$$\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{[t_k^n, t_k^n+1[}(\sigma) \sum_{\{i>k; t_i^n - t_k^n \leq \delta\}} ((\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n])^2 - (\mathbf{E}[A_{t_{i-1}^n} | \mathcal{F}_{t_{i-1}^n}^n])^2)]$$

$$\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{[t_k^n, t_k^n+1[}(\sigma)(A^2_{t_{k+1}^n+\delta} - (A^n_{t_k^n})^2)]$$

$$\leq \sum_{p=1}^{q(\delta)} \mathbf{E}[1_{[s_p, s_{p+1}[}(\sigma)(A^2_{s_{p+1}+2\delta} - (A^n_{s_p-\delta})^2)]$$

(where s_p with $0 \leq p \leq q(\delta)$ are the points of a subdivision of interval $[0, T]$ whose mesh is lower than δ),

$$\leq \sum_{p=1}^{q(\delta)} \mathbf{E}[1_{[s_p, s_{p+1}[}(\sigma)(A^2_{s_{p+1}+2\delta} - A^2_{s_p})] + \varepsilon'$$

(as soon as n is large enough)

$$\leq \mathbf{E}[A^2_{\sigma+2\delta} - A^2_{\sigma-\delta}] + \varepsilon' \leq 2\varepsilon'$$

for δ small enough.

Hence we have the following, which proves tightness of (M^n) :

For every $\varepsilon > 0$, for every $\alpha > 0$, there exists δ_0, such that for every $\delta \leq \delta_0$

$$\limsup_n \sup_{\{(\sigma,\tau); \sigma \leq \tau \leq \sigma+\delta\}} \mathbf{P}[|M_\tau^n - M_\sigma^n| > \alpha] \leq \varepsilon.$$

Writing now for every $\mathcal{F}^n$-stopping time σ

$$\mathbf{P}[|\Delta M_\sigma^n| > \alpha] \leq \frac{1}{\alpha^2} \mathbf{E}[(\Delta M_\sigma^n)^2].$$

Then, exactly as above:

$$\mathbf{E}[(\Delta M_\sigma^n)^2] = \mathbf{E}[\sum_{\sigma = t_k^n} (\Delta M_{t_i^n}^n)^2]$$

$$= \mathbf{E}[\sum_{\sigma = t_k^n} (\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n] - \mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2]$$

$$\leq \sum_{k=0}^{k_n} \mathbf{E}[1_{\{t_k^n\}}(\sigma)(\mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_i^n}^n] - \mathbf{E}[A_{t_i^n} | \mathcal{F}_{t_{i-1}^n}^n])^2].$$

Using the same tools we get finally, considering $\sigma = \inf\{t : |\Delta M_t^n| > \varepsilon\}$: for every ε,

$$\mathbf{P}[\sup_{t\leq T} |\Delta M_t^n| > \varepsilon] \xrightarrow{\mathbf{P}} 0$$

hence the desired result ([5], chapt.VI).

3) Last step. The sequence $((X, M^n, Y))$ is tight in $\mathbb{C} \times \mathbb{D}^2$. By representation Theorem of Skorokhod, we can find, on a suitable space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$ a sequence $((\bar{X}^n, \bar{M}^n, \bar{Y}^n))$ relatively compact for the convergence in probability, and for a subsequence (indexed also by n), we have :

$$(\bar{X}^n, \bar{M}^n, \bar{Y}^n) \xrightarrow{\bar{\mathbf{P}}})(\bar{X}, \bar{M}, \bar{Y})$$

where $\mathcal{L}((\bar{X}^n, \bar{M}^n, \bar{Y}^n)|\bar{P}) = \mathcal{L}((X, M^n, Y)|\mathbf{P})$ and $\mathcal{L}((\bar{X}, \bar{Y})|\bar{P}) = \mathcal{L}((X, Y)|\mathbf{P})$.

Let us consider $\hat{Y}^n$ the step process of order n of $\bar{Y}^n$, and denote $\hat{X}^n = \bar{\mathbf{E}}[\bar{X}^n|\mathcal{F}^{\hat{Y}^n}]$; we have :

$$\mathcal{L}((\bar{X}^n, \hat{X}^n, \bar{M}^n, \bar{Y}^n, \hat{Y}^n)|\bar{P}) = \mathcal{L}((X, X^n, M^n, Y, Y^n)|\mathbf{P}).$$

Let us denote $\tilde{A}^n = \hat{X}^n - \bar{M}^n$; then $\mathcal{L}((\tilde{A}^n, \bar{M}^n)|\bar{P}) = \mathcal{L}((A^n, M^n)|\mathbf{P})$.

We have (for almost all t) the convergence $\hat{X}_t^n \to \bar{X}_t$ in L^1.

Actually,

$$\bar{\mathbf{E}}[|\hat{X}_t^n - \bar{X}_t|] \leq \bar{\mathbf{E}}[|\hat{X}_t^n - \bar{X}_t^n|] + \bar{\mathbf{E}}[|\bar{X}_t^n - \bar{X}_t|].$$

We have $\bar{\mathbf{E}}[|\hat{X}_t^n - \bar{X}^n|] = \mathbf{E}[|\mathbf{E}[X|\mathcal{F}_t^n] - X_t|]$ and this expression converges to 0 for $n \to \infty$.

On the other hand, $\bar{X}_t^n \xrightarrow{\bar{\mathbf{P}}} \bar{X}_t$ and the sequence $(\bar{X}_t^n)$ is bounded in L^2.

Finally we get that, for almost every t, $\tilde{A}_t^n \xrightarrow{\bar{\mathbf{P}}} \bar{X}_t - \bar{M}_t$. $\bar{X} - \bar{M}$ is then a continuous increasing process $\tilde{A}$, the convergence is then uniform in t ; we deduce that $\bar{M}$ which is a continuous martingale and also difference of two increasing processes is 0. The proof is complete.

Remark One can deduce the infinitesimality of jumps of M^n, from the Aldous work [1]. Since (X^n) is a sequence of $\mathcal{F}^n$-submartingale, that (X_T^n) is uniformly integrable and X continuous, we get : for every ε, $\limsup_n \mathbf{P}[\sup_{s\leq T}(X_s^n - X_s) > \varepsilon] = 0$. We deduce immediately $\limsup_n \mathbf{P}[\sup_{s\leq T} \Delta M_s^n > 2\varepsilon] = 0$, and under the martingality of M^n, $\limsup_n \mathbf{P}[\sup_{s\leq T} |\Delta M_s^n| > \varepsilon] = 0$. This last deduction holds because of predictable character of jumps of M^n, it would be wrong if M^n was, for example, quasileftcontinuous.

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