On the countermeasures to the higher genus torsion point attacks on SIDH

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IRMAR, Rennes, December 2nd, 2022

Outline

Generalities and SIDH

Key agreement

Two parties : Alice (red) and Bob (blue)

Aim: share the same key (bit string, integer, ...)

Obstacle: they are far away from each other, internet is not safe.

Solution: Diffie-Hellman key agreement.
Both parties agree on group $G = \langle g \rangle$ of prime order p.

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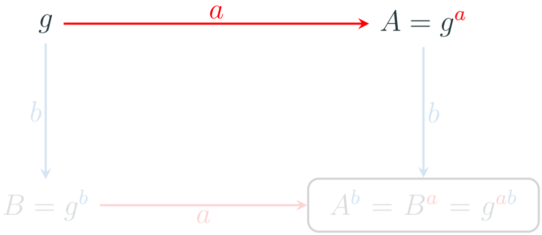

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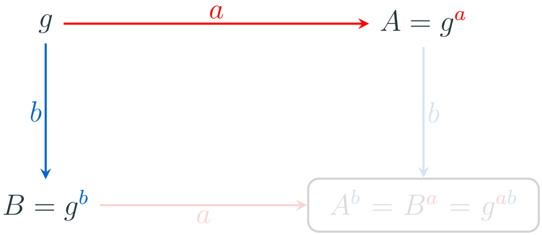

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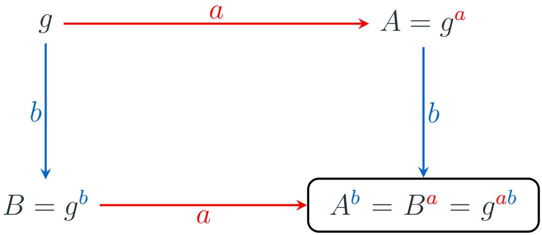

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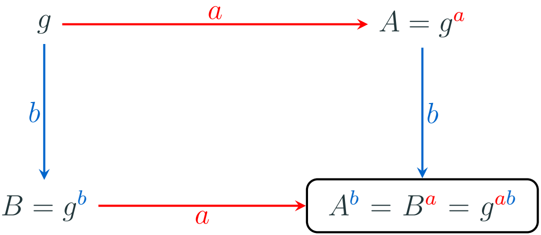

Break the protocol: recover the shared key by spying on the internet (or chanel).

CDH: Given g, p, $A = g^a$ and $B = g^b$, find g^{ab}.

DL: Given g, p and $A = g^a$, find a.

Hard to break using classical computer

Easy to break with quantum computer

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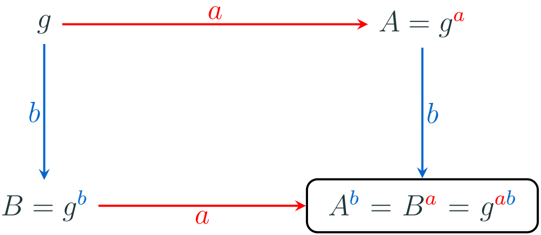

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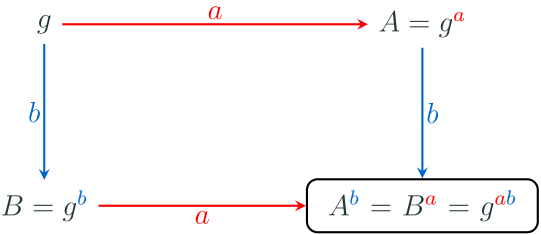

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Post-Quantum: hard for both classical and quantum computers.

Lattices, Codes, **Isogenies**, Multivariate equations, Hash Functions, ...

Isogeny-based Cryptography:

- Very compact keys
- Offers a good replacement for Diffie-Hellman (NIKE)

But:

- Relatively slow
- Young field

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Elliptic curves: $E : y^2 = x^3 + Ax + B$, are abelian groups.

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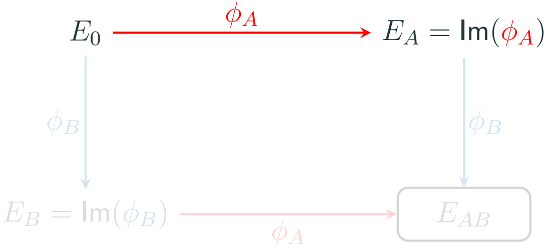

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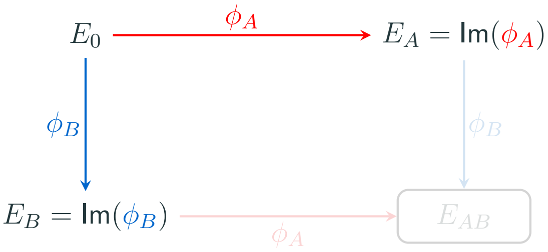

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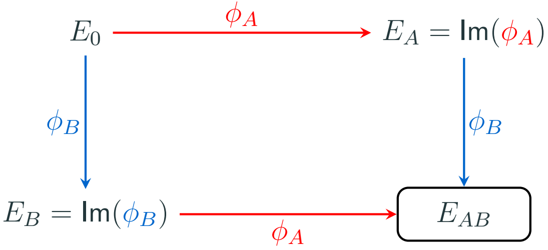

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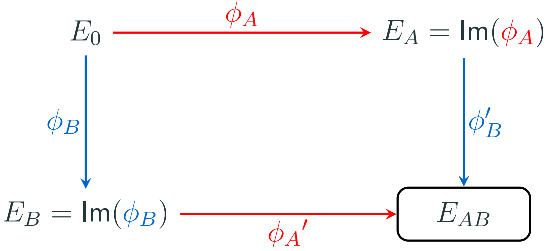

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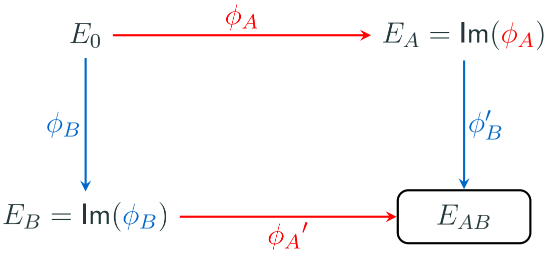
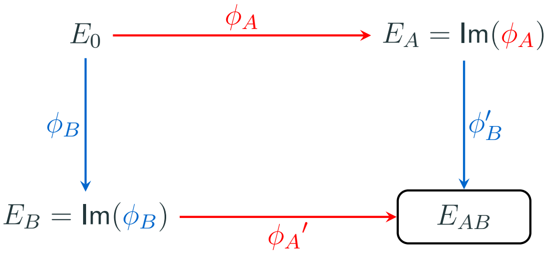


Commutativity !!: use ordinary isogenies $\rightarrow$ CRS[1].

1. Inefficient
2. Quantum sub-exponential time (group actions)

[1] Couveignes-Rostotsev-Stulbunov 1996/2006



Efficient and no quantum attack !!: use supersingular isogenies.

1. Do not commute !!

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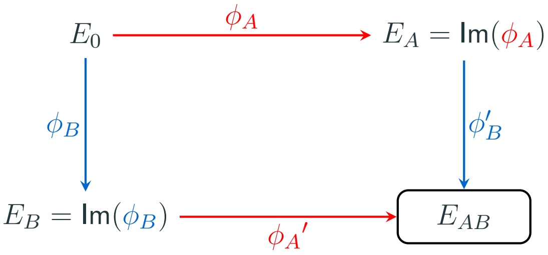

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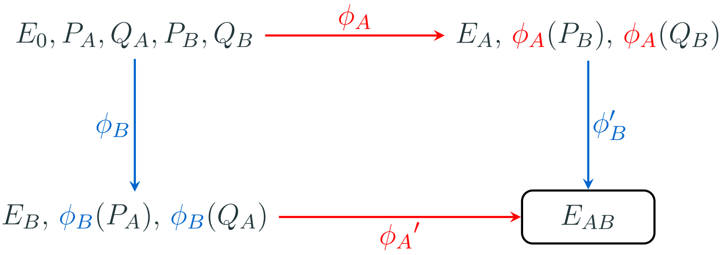

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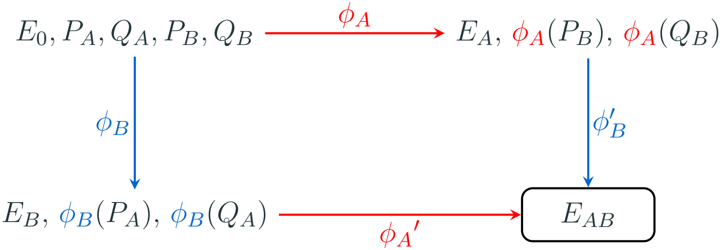
Ambient field: $\mathbb{F}_{p^2}$, $p = 2^a 3^b - 1$. $\quad \deg \phi_A = 2^a \quad \deg \phi_B = 3^b$

$E_0[2^a] = \langle P_A, Q_A \rangle, \quad E_0[3^b] = \langle P_B, Q_B \rangle$

Diffie-Hellman with isogenies



SSI-CDH: Given E_0, P_A, Q_A, P_B, Q_B, E_A, $\phi_A(P_B)$, $\phi_A(Q_B)$, E_B, $\phi_B(P_A)$ and $\phi_B(Q_A)$, compute E_{AB}.

SSI-T: Given E_0, P_A, Q_A, P_B, Q_B, E_B, $\phi_B(P_A)$ and $\phi_B(Q_A)$, compute ϕ_B.

SIDH's life span

Life was nice till 2016: year where a demon possessed the TP!

GPST 2016: adaptive attack on SIDH, only countered by the FO transform

Petit 2017: torsion point attack on imbalanced SIDH, no impact on SIDH

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CD-MM-R 2022, final shot: SIDH/SIKE is broken in seconds...

All these attacks exploit torsion point information !!

Non exhaustive list: BdQL+ 2019, ...

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Countermeasures

CD-MM-R attacks require:

1. torsion points information;
2. degree of the secret isogeny.

Two countermeasures:

- Masked-degree SIDH (MD-SIDH): the degree of the secret isogeny is secret;
- Masked torsion points SIDH (M-SIDH): the degree of the secret isogeny if fixed, but the torsion point images are scaled by a secret scalar.

Current analysis: field characteristic $\log_2 p \approx 6000$, as oppose to $\log_2 p \approx 434$ in SIDH, for 128 bits of security.

Torsion point attacks

More facts about isogenies

$E/\mathbb{F}_q$: n-torsion group ($p \nmid n$)

$$E[n] = \langle P, Q \rangle \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$$

Supersingular curves:

- $\mathrm{End}(E) \simeq \mathcal{O}_{\max} \subset \mathcal{B}_{p,\infty}$
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Dual d-isogeny: $\varphi : E \to E' \iff \exists!^* \hat{\varphi} : E' \to E,$ such that $\hat{\varphi} \circ \varphi = [d]_E$ and $\varphi \circ \hat{\varphi} = [d]_{E'}$.

We have

$$\ker \hat{\varphi} = \varphi(E[d]) \quad \text{and} \quad \ker \varphi = \hat{\varphi}(E'[d]).$$

Pairings and isogenies: $\phi : E \longrightarrow E'$, $E[N] = \langle P, Q \rangle$, then

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The framework

SSI-T Problem: Given E_0, $E[B] = \langle P, Q \rangle$, E, $\phi(P)$, $\phi(Q)$, compute ϕ.

Degree transformation: define a map Γ that can be used to transform ϕ to $\tau = \Gamma(\phi, input)$ such that:

1. Knowing $\tau = \Gamma(\phi, input)$, one can recover ϕ
2. τ can be evaluated on the B-torsion
3. τ can be recovered from its action on the B-torsion

The attack: Given a suitable description of Γ,

- Use 2. and 3. to recover τ
- Use 1. to derive ϕ from τ

Assumes that End(E_0) is known. $input = [\theta \in End(E_0),\, d \in \mathbb{Z}]$.

$$\tau = \Gamma(\phi, \theta, d) := [d] + \phi \circ \theta \circ \hat{\phi}$$

s.t. $\deg \tau = B^2 e$ with e small.

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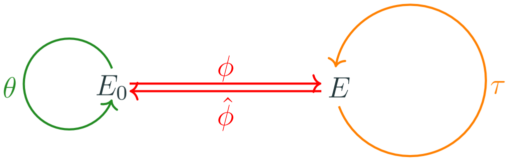

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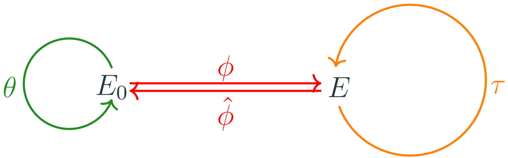

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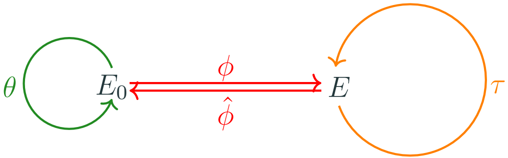

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$\ker \hat{\phi} =^* \ker(\tau - [d]) \cap E[A]$

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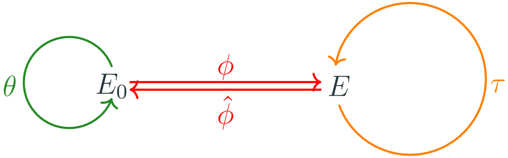

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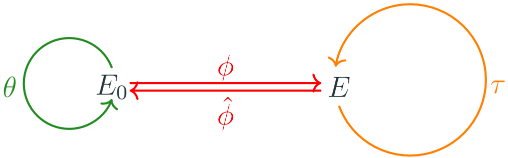

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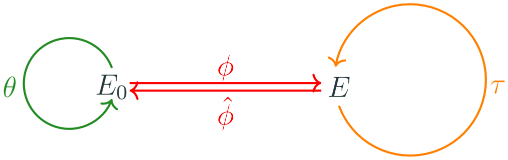

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Requires: $B > pA$; while in SIDH $A \approx B \approx \sqrt{p}$.

Assume $\phi : E_0 \longrightarrow E_B$ has degree B and the TP have order A.
Set $a = A - B = a_1^2 + a_2^2 + a_3^2 + a_4^2$.

$$\tau = \Gamma(\phi, a) := \begin{bmatrix} \alpha_0 & \hat{\phi} Id_4 \\ -\phi Id_4 & \hat{\alpha}_B \end{bmatrix} \in \mathrm{End}(E_0^4 \times E_B^4)$$

where

- $\phi Id_4 : E_0^4 \longrightarrow E_B^4$ and $\hat{\phi} Id_4 : E_B^4 \longrightarrow E_0^4$
- $\alpha_0 \in \mathrm{End}(E_0^4)$ and $\alpha_B \in \mathrm{End}(E_B^4)$ having the same matrix representation

$$M = \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{bmatrix}$$

Assume $\phi : E_0 \longrightarrow E_B$ has degree B and the TP have order A. Set $a = A - B = a_1^2 + a_2^2 + a_3^2 + a_4^2$.

$$\tau = \Gamma(\phi, a) := \begin{bmatrix} \alpha_0 & \hat{\phi} Id_4 \\ -\phi Id_4 & \hat{\alpha}_B \end{bmatrix} \in \mathrm{End}(E_0^4 \times E_B^4)$$

Fact: τ has degree $B + a = A$

1. Knowing $\tau = \Gamma(\phi, input)$, one can recover ϕ
2. τ can be evaluated on the A-torsion
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CD-MM-R 2022

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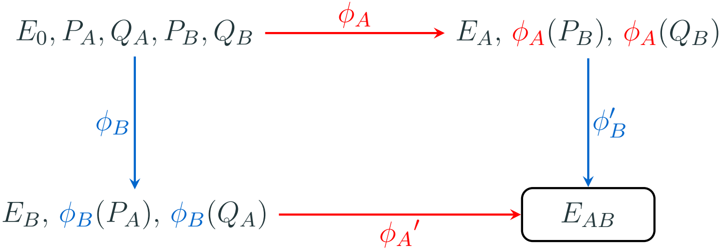
Runs in polynomial time when $A^2 > B$!!
Breaks SIDH/SIKE/SETA/...

Countermeasures

Masked degree SIDH



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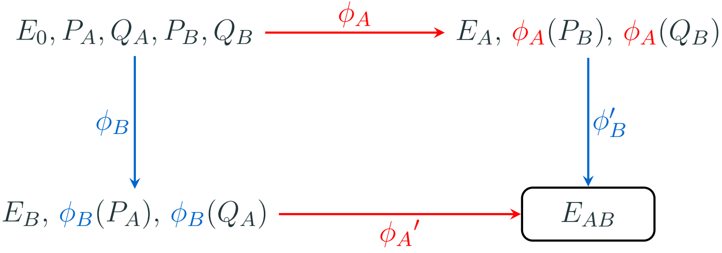

Ambient field: $\mathbb{F}_{p^2}$, $p = \ell_1^{a_1} \cdots \ell_t^{a_t} q_1^{b_1} \cdots q_t^{b_t} f - 1$

$A := \prod_{i=1}^{t} \ell_i^{a_i} \qquad B := \prod_{i=1}^{t} q_i^{b_i}, \quad A \approx B.$

$\deg \phi_A = A', \quad A' | A, \qquad \deg \phi_B = B', \quad B' | B.$

$E_0[A] = \langle P_A, Q_A \rangle, \quad E_0[B] = \langle P_B, Q_B \rangle$

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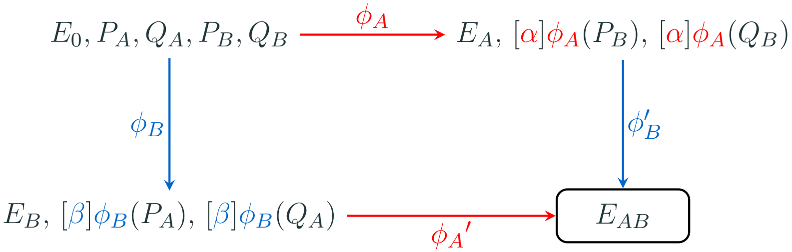

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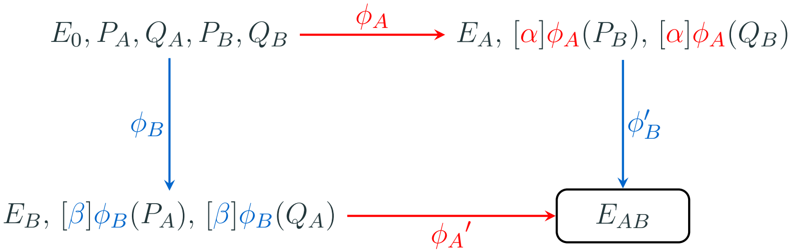
Hide the degree from pairings: $\alpha \in (\mathbb{Z}/B\mathbb{Z})^\times \quad \beta \in (\mathbb{Z}/A\mathbb{Z})^\times$

Masked torsion points SIDH



Ambient field: $\mathbb{F}_{p^2}$, $p = \ell_1 \cdots \ell_\lambda q_1 \cdots q_\lambda f - 1$

$A := \prod_{i=1}^{\lambda} \ell_i \qquad B := \prod_{i=1}^{\lambda} q_i, \quad A \approx B.$

$\deg \phi_A = A, \qquad \deg \phi_B = B.$

$E_0[A] = \langle P_A, Q_A \rangle, \quad E_0[B] = \langle P_B, Q_B \rangle$

Hide the exact TP images: $\alpha \in \mu_2(\mathbb{Z}/B\mathbb{Z}) \quad \beta \in \mu_2(\mathbb{Z}/A\mathbb{Z})$

Analysis of the countermeasures

Case of M-SIDH: using less torsion

CD-MM-R attack : works when $A^2 > B$.

In M-SIDH, $A \approx B = (\sqrt{B})^2$.

Hence we can use less torsion $B' = \prod_{i=t}^{\lambda} \ell_i > \sqrt{B}$.

Guessing the exact torsion point: $O(2^{\lambda - t})$

Consequence: A and B must have at least 2λ distinct prime factors each.

Case of M-SIDH: using lollipop endomorphisms

Given a small $\theta \in End(E_0)$, eliminate the scalar β in M-SIDH:

With respect to the A torsion, we have:

$$([\beta]\phi_B) \circ \theta \circ (\widehat{[\beta]\phi_B}) = [\beta^2] \circ \phi_B \circ \theta \circ \widehat{\phi_B} \equiv \phi_B \circ \theta \circ \widehat{\phi_B} =: \tau.$$

$\deg \tau = B^2 \deg \theta$.

CD-MM-R on τ requires : $\sqrt{\deg \tau} = B\sqrt{\deg \theta} \approx B$ (for small θ).

Consequence: No small endomorphisms in E_0, if possible, no known endomorphism at all.

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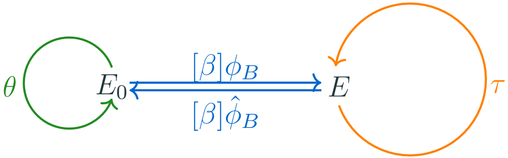

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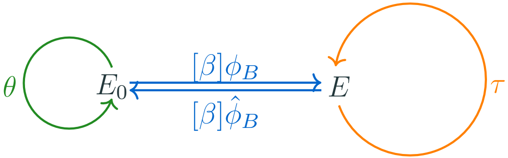

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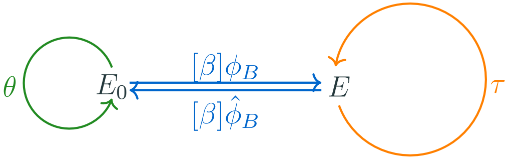

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Case of MD-SIDH: recovering the square free part

Recall: $\deg \phi_B = B'|B$, TP are scaled by $\beta \in \mathbb{Z}/B\mathbb{Z}$.

Pairings are used to recover $\beta^2 B' \mod A$. Define:

$$\begin{array}{rccl} \chi_i : & (\mathbb{Z}/\ell_i^{a_i}\mathbb{Z})^\times & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & x & \longmapsto & \begin{cases} 1 & \text{if x is a quad. residue modulo } \ell_i^{b_i}; \\ 0 & \text{if not.} \end{cases} \end{array}$$

$$\begin{array}{rccc} \Phi : & D(q_1 \cdots q_t) & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^t \\ & N & \longmapsto & (\chi_1(N), \ldots, \chi_t(N)) \end{array}$$

Claims:

- We can evaluate Φ on the square free part of B'
- Φ is almost injective.

Consequence: We can recover the square free part B'_1 of B'.

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Assume that we know B'_1. Set $B_0 = \max\{n \mid n|B, n^2 B'_1 \leq B\}$.
Then $\exists \beta_0$, divisor of B, $N_B := B_0^2 B'_1 = \beta_0^2 B' \leq B$.

Set $\phi_0 = [\beta_0] \circ \phi_B$, then $\deg(\phi_0) = N_B$ is known.

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Adaptive security and parameters size

- GPST and the F-Petit adaptive attacks on M-SIDH: straightforward.
- FP adaptive attack on MD-SIDH: uses the reduction of MD-SIDH to M-SIDH.
- GPST on MD-SIDH: not straightforward, but possible.

Parameter selection:

- $n|B$, $n > \sqrt{B} \quad \longrightarrow \quad \lambda$ odd prime factors.
- $End(E_0)$ unknown

AES	NIST	p (in bits)	secret key	public key
128	level 1	5911	≈ 369 bytes	4434 bytes
192	level 3	9382	≈ 586 bytes	7037 bytes
256	level 5	13000	≈ 812 bytes	9750 bytes

On the claims of eprint 2022/1667

Two days ago on eprint : Applying Castryck-Decru Attack on the Masked Torsion Point Images SIDH variant

Successfully applies CD attack on M-SIDH with SIDH primes.

Claims that it will also be successfull with M-SIDH primes.

Success rate of CD attack on M-SIDH with SIDH primes:
Expected : 1/2   Observed : 1.

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But today, they killed SIDH.

Two countermeasure ideas were suggested and analysed:
M-SIDH and MD-SIDH.

Outcome of the analysis: field characteristic must be at least ≈ 6000 bits !

Still vulnerable to adaptive attacks. Require FO to achieve IND-CCA security.

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Happy to discuss your comments and questions !!!