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Identification of an isometric transformation of the standard Brownian sheet

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Abstract

We use *quadratic* variations to identify almost surely the coordinate system where the standard Brownian sheet is defined. This identification is carried out with the help of an algorithmic-like procedure.

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Keywords: Coordinate system; Estimation; Quadratic variation

1. Introduction

Random fields, or spatial processes, are useful for modeling spatial phenomenon in environmental sciences, hydrology, geophysics, medical or industrial images (cf. Sampson and Guttorp (1992) and its bibliography).

The analysis, modeling and estimation, of the spatial structure has been the subject of increasing research. Knowledge of the spatial covariance is fundamental in spatial estimation or kriging. As outlined in (Sampson et al., 2001), particularly important is the fact that the

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underlying processes are almost always non-stationary, neither isotropic, over the spatial scales of interest.

In practice, when we model a non-stationary or/and a non-isotropic spatial phenomenon with a random field, the coordinate system where the latter is defined cannot be chosen arbitrarily. In this case, the identification of the coordinate system is as important as the estimation of the parameters of the random field itself.

In some practical situations, a privileged orthogonal frame is intrinsically linked to the data: for instance, the crystalline lamellar silicates (Colliex et al., 1980); the grain boundary density of a deformed ferritic steel (Ohser and Mucklich (2000)). In some cases, the identification of this privileged frame is very important. For example, the trabecular structure of bones has to be clearly identified from X-ray images to detect ill (osteoporotic) bones. Once the true axes are detected some works have been developed for the diagnosis of osteoporosis (cf. Léger, 2000; Jennane et al., 2001; Lemineur et al., 2004). The trabecular structure is related to horizontal and vertical directions. In this orthogonal frame, it has been shown that the trabecular architecture is well modeled by rectangular increments of a Gaussian random sheet.

In this paper, we study this problem for random fields indexed by a continuous spatial parameter in $\mathbb{R}^2$ observed on a fine two-dimensional regular grid. In this context, the asymptotic setup we consider is infill asymptotics where the samples are taken from a fixed bounded region and where the sampling locations become increasingly dense. This asymptotic setup is particularly well adapted for high-resolution data, such that satellite data Matthews (1983) or X-ray images are used for the diagnosis of osteoporosis Bonami and Estrade (2003).

The standard Brownian sheet is an illustration of such random fields whose definition depends on the coordinate system. We propose an estimation of this coordinate system by using quadratic variations. First, we identify the quadrant where the Brownian sheet is observed. Then we give a practical procedure to estimate the coordinate system. Remark that the Brownian sheet is zero variance along the (unknown) coordinate axes. Even though we can doubt the realism of such a model, this model is an academic case which makes our method explicit. Moreover, this zero variance property has no importance in the sense that our method does not depend on it.

The quadratic variations are first introduced by Lévy (1940) who shows that if B is the standard Brownian process on [0,1], then, almost surely, its quadratic variation on [0,1] converges to 1. Baxter (1956) and further Gladyshev (1961) generalize this result to a large class of Gaussian processes. Guyon and León (1989) introduce the H-variations for stationary Gaussian processes, a generalization of these quadratic variations. They study the convergence in distribution of the H-variations, suitably normalized.

For Gaussian process Z with stationary increments, Istas and Lang (1997) define generalized quadratic variations, substituting a general discrete difference operator to the simple first-order difference $Z(k/n)-Z((k-1)/n)$. They use these quadratic variations to estimate the Hölder index of a process.

The generalization of quadratic variations for stationary Gaussian fields indexed by $\mathbb{R}^2$ is studied in Guyon (1987) as well as in León and Ortega (1989). Another generalization for non-stationary Gaussian processes over general index spaces and quadratic variations along curves is done in Adler and Pyke (1993). Note that in the d-dimensional context ($d>1$), several families of variations (linear, superficial, product) are available which provide useful tools for identifying different models: for example, Orstein–Uhlenbeck processes can be identified in mean square on $\mathbb{R}^2$, but not on $\mathbb{R}$ (Guyon, 1987).

The paper is structured as follows: Section 2 sets up notations, assumptions and definitions. Section 3 describes the quadratic variations and their asymptotic properties. In Section 4, these quadratic variations are combined to propose almost sure consistent estimators of the parameters of the isometric transformation. In Section 5, simulations evaluate the performance of these estimators. Finally, Section 6 discusses extensions of the present work and further topics for further research.

2. Notations, assumptions and definitions

First note that angles we are dealing with are oriented. Let $W = \{W(x, y), (x, y) \in \mathbb{R}^2\}$ be the standard Brownian sheet, that isa centered Gaussian process indexed by $\mathbb{R}^2$ with a covariance function given by

$$E(W(x_1, y_1)W(x_2, y_2)) = \tfrac{1}{4}(|x_1| + |x_2| - |x_1 - x_2|)(|y_1| + |y_2| - |y_1 - y_2|). \quad (1)$$

This definition is given in a certain coordinate system (we name it, in the sequel, the canonical coordinate system or canonical axes), say $(O', \mathscr{X}, \mathscr{Y})$. Suppose that instead of $W(x, y)$, we observe $Y_{\theta,a,b}(u, v) = W(x(u, v), y(u, v))$, where

$$x(u, v) = (u - a)\cos(\theta) + (v - b)\sin(\theta), \quad (2)$$

$$y(u, v) = -(u - a)\sin(\theta) + (v - b)\cos(\theta). \quad (3)$$

The covariance function of $Y_{\theta,a,b}$ is given by (1)–(3). In practice, this means that we model a phenomenon with a standard Brownian sheet for which we do not know the coordinate system.

As illustrated in Fig. 1, W is defined in the unknown coordinate system $(O', \mathscr{X}, \mathscr{Y})$ and is observed through $Y_{\theta,a,b}$ in a known and fixed coordinate system $(O, \mathscr{U}, \mathscr{V})$. The direct isometry from $(O, \mathscr{U}, \mathscr{V})$ to $(O', \mathscr{X}, \mathscr{Y})$ is a rotation through an angle θ about the origin O, followed by a translation of vector (a, b).

In this paper, we are interested in the estimation of (θ, a, b) from one realization of $Y_{\theta,a,b}$ observed on $\mathscr{B}$, some non-vacuous open subset of $\mathbb{R}^2$. In fact, because $Y_{\theta,a,b}$ is null on the canonical axes, we assume that $\mathscr{B}$ does not intersect any canonical axis. There are two reasons for this. First, the analytical nature of our identification method allows us to observe $Y_{\theta,a,b}$ on a open set as small as we want, so that it is reasonable to assume that the observation set $\mathscr{B}$ does not intersect the canonical axes. With this method, it could thus be possible to identify a bijective transformation of class C^1. In Guyon and Perrin (2000), such a bijection is identified in the context of a Gaussian random field with separable covariance. Second, we work in a less informative context where we do not have the information corresponding to the points of the canonical axes, where $Y_{\theta,a,b}$ is almost surely null.

The estimation is carried out with the help of quadratic variations on various segments. Results from Baxter (1956) and its generalizations in Perrin (1999) are widely used.

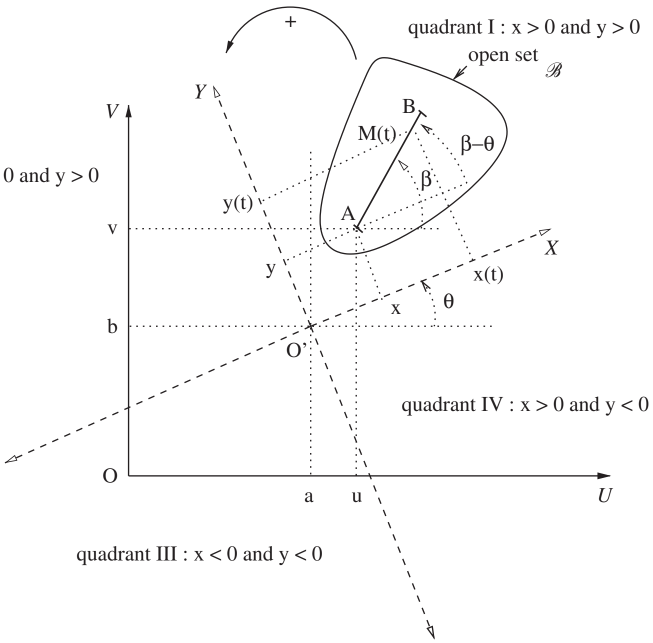

Fig. 1. Isometric transformation of the coordinate system, $\mathscr{B}$ is the observation domain of $Y_{\theta,a,b}$.

3. Quadratic variations

We only consider quadratic variations along segments, like the segment $[A, B] \subset \mathscr{B}$ of length $L > 0$ (see Fig. 1), where A is the point of coordinates (u, v) in $(O, \mathscr{U}, \mathscr{V})$. Let β denote the angle between $\vec{Ou}$ and $\vec{AB}$. Each point $M(t) = (x(t), y(t))$ belonging to $[A, B]$ has the following parametric representation in $(O', \mathscr{X}, \mathscr{Y})$, for $t \in [0, 1]$:

$$x(t) = (u - a)\cos(\theta) + (v - b)\sin(\theta) + Lt\ \cos(\beta - \theta),$$
$$y(t) = -(u - a)\sin(\theta) + (v - b)\cos(\theta) + Lt\ \sin(\beta - \theta).$$

Without restriction, we may impose $\theta \in [0, \pi/2[$. Unlike θ, the angle β is a parameter under our control, like L, u and v, and we choose β in $[0, \pi/2]$. The restriction of W to the segment $[A, B]$ is the process Z indexed by [0,1]

$$Z(t) = W(x(t), y(t)). \tag{4}$$

The coordinates of A in $(O', \mathscr{X}, \mathscr{Y})$ are $x = x(0)$ and $y = y(0)$.

In Section 3.1, we define the quadratic variations along the segment $[A, B]$. As we will see in Section 3.2, the behavior of these quadratic variations will depend on the quadrant of $(O', \mathscr{X}, \mathscr{Y})$ in which Y is observed.

3.1. Results in the positive quadrant

In a first step, we assume that $x>0$ and $y>0$ and that $[A, B]$ is included in the positive quadrant (quadrant I of Fig. 1) and that it does not meet its axes. We consider the three other cases ($x<0$ and $y>0$ (quadrant II), $x<0$ and $y<0$ (quadrant III), and $x>0$ and $y<0$ (quadrant IV)) in Section 3.2.

Let n be a positive integer. We set for $k=1,2,\ldots,n$

$$\Delta Z_k = Z(k/n) - Z((k-1)/n).$$

Let $\Pi_n(1)=\{0,1/n,2/n,\ldots,(n-1)/n,1\}$ be the regular partition of [0,1] at constant scale $1/n$. Think of $1/n$ as the degree of resolution of an image: the finer resolution ($n\to\infty$) the closer to its limit. Let $[nt]$ be the greatest integer smaller than or equal to nt. For $t\in]0,1]$ and $[nt]\geqslant 1$, we define the quadratic variations $V_n(t;\beta,u,v,L)$ of Z along $\Pi_n(t)=\{0,1/n,2/n,\ldots,[nt]/n\}$ as follows:

$$V_n(t;\beta,u,v,L)=\sum_{k=1}^{[nt]}(\Delta Z_k)^2.$$

Before giving the main results concerning $V_n(t;\beta,u,v,L)$, we must check that the covariance function $r(t,t')=E(Z(t)Z(t'))$ of Z is continuous in $[0,1]^2$ and has second derivatives, which are uniformly bounded for $t\neq t'$.

From (1) and (4) we have, for any t, t' in $[0,1]$

$$\begin{aligned} r(t,t') &= (x+Lt\cos(\beta-\theta))\wedge(x+Lt'\cos(\beta-\theta))(y+Lt\sin(\beta-\theta)) \\ &\quad \wedge(y+Lt'\sin(\beta-\theta)). \end{aligned}$$

We denote using $r^{(m,m')}(t,t')$ the m,m'-partial derivative of r with respect to t and t'. Depending on the sign of $\sin(\beta-\theta)$, that is on the sign of $\beta-\theta$, we have to consider two situations (S1) and (S2).

(S1) $0\leqslant\beta-\theta\leqslant\pi/2$, then $r(t,t')=(x+L(t\wedge t')\cos(\beta-\theta))(y+L(t\wedge t')\sin(\beta-\theta))$ so that

- for $t'>t$, $r^{(1,1)}(t,t')=r^{(0,2)}(t,t')=0$, $r^{(2,0)}(t,t')=L^2\sin(2(\beta-\theta))$ and

$$r^{(0,1)}(t,t')=0. \tag{5}$$

- for $t'<t$, $r^{(1,1)}(t,t')=r^{(2,0)}(t,t')=0$, $r^{(0,2)}(t,t')=L^2\sin(2(\beta-\theta))$ and

$$\begin{aligned} r^{(0,1)}(t,t') &= L(u-a)\sin(\beta-2\theta)+L(v-b)\cos(\beta-2\theta) \\ &\quad + L^2t'\sin(2(\beta-\theta)). \end{aligned} \tag{6}$$

(S2) $-\pi/2 \leqslant \beta-\theta<0$, then $r(t,t')=(x+L(t\wedge t')\cos(\beta-\theta))(y+L(t\vee t')\sin(\beta-\theta))$ so that

- for $t'>t$, $r^{(0,2)}(t,t')=r^{(2,0)}(t,t')$, $r^{(1,1)}(t,t')=\frac{L^2\ \sin(2(\beta-\theta))}{2}$ and

$$r^{(0,1)}(t,t')=L((u-a)\cos(\theta)+(v-b)\sin(\theta))\sin(\beta-\theta) + \frac{L^2 t\ \sin(2(\beta-\theta))}{2}. \tag{7}$$

- for $t'<t$, $r^{(0,2)}(t,t')=r^{(2,0)}(t,t')$, $r^{(1,1)}(t,t')=\frac{L^2\ \sin(2(\beta-\theta))}{2}$ and

$$r^{(0,1)}(t,t')=L(-(u-a)\sin(\theta)+(v-b)\cos(\theta))\cos(\beta-\theta) \times \frac{L^2 t\ \sin(2(\beta-\theta))}{2}. \tag{8}$$

Finally, in both situations, $r(t,t')$ is continuous in $[0,1]^2$ and has second derivatives which are uniformly bounded for $t\neq t'$.

Definition 3.1. Define the singularity function α_i of Z in situation (Si); $i=1,2$, for $t\in[0,1]$ by

$$\alpha_i(t;\beta,u,v,L)=\lim_{t'\nearrow t} r^{(0,1)}(t,t')-\lim_{t'\searrow t} r^{(0,1)}(t,t').$$

This singularity function represents the discontinuity of the first partial derivatives of the covariance on the diagonal. This discontinuity locally defines the behavior of the quadratic variation. From (5)–(8) we obtain

$$\alpha_1(t;\beta,u,v,L)=L(u-a)\sin(\beta-2\theta) + L(v-b)\cos(\beta-2\theta)+L^2 t\ \sin(2\beta-2\theta), \tag{9}$$

$$\alpha_2(t;\beta,u,v,L)=-L(u-a)\sin(\beta)+L(v-b)\cos(\beta). \tag{10}$$

This difference between α_1, linear in t, and α_2, constant in t, is explained in Fig. 2 where the infinitesimal variation ΔZ of Z is represented in each situation.

The next Theorem 3.1 is a direct consequence of the following result in Perrin (1999) (Theorem 5.1).

Lemma 3.1. *Let $Z=\{Z(t),\ t\in[0,1]\}$ be a real-valued centered Gaussian random process with covariance function r and singularity function α. Assume that r is continuous in* $[0,1]^2$ *and has second derivatives which are uniformly bounded for $t\neq t'$, and that α has a bounded first derivative in* $[0,1]$. *Then, almost surely*

$$\lim_{n\to\infty}\sup_{t\in[0,1]}\left|\sum_{k=1}^{[nt]}(\Delta Z_k)^2-\int_0^t \alpha(w)\,\mathrm{d}w\right|=0.$$

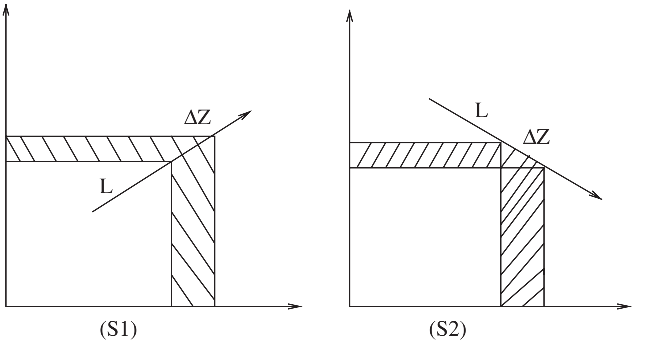

Fig. 2. Variation of Z in (S1) and (S2).

For t in [0, 1], let $V_{n,i}(t;\beta,u,v,L)$ be the quadratic variation $V_n(t;\beta,u,v,L)$ in situation (Si), $i=1,2$. We can now directly deduce the following result from Lemma (3.1):

Theorem 3.1. *For $i=1,2$, almost surely*

$$\lim_{n\to\infty}\sup_{t\in[0,1]}\left|V_{n,i}(t;\beta,u,v,L)-\int_0^t \alpha_i(w;\beta,u,v,L)\,\mathrm{d}w\right|=0.$$

We set

$$\begin{aligned}I_1(t;u,v,L,\beta)&=\int_0^t \alpha_1(w;\beta,u,v,L)\,\mathrm{d}w\\&=((u-a)\sin(\beta-2\theta)+(v-b)\cos(\beta-2\theta))Lt\\&\quad+\frac{t^2L^2}{2}\sin(2\beta-2\theta)\end{aligned}\tag{11}$$

and

$$\begin{aligned}I_2(t;u,v,L,\beta)&=\int_0^t \alpha_2(w;\beta,u,v,L)\,\mathrm{d}w\\&=(-(u-a)\sin(\beta)+(v-b)\cos(\beta))Lt.\end{aligned}\tag{12}$$

3.2. Results in the other quadrants

Similar computations lead to results we summarize in Table 1 where, for instance, (I,S1) means quadrant I and situation (S1), the quadrants being defined in Fig. 1.

4. Estimation of the transformation

Let us first recall that L, u, v and β are the parameters under our control and that a, b and θ are the parameters we want to estimate. For this purpose, we assume that $Y_{\theta,a,b}$ is observed in $\mathscr{B}$, a subset included in one of the four quadrants of Fig. 1.

Table 1
Singularity function in each quadrant

Quadrant × Situation	Singularity function
(I,S1) and (II,S2)	$L(u-a)\sin(\beta-2\theta)+L(v-b)\cos(\beta-2\theta)+L^2t\ \sin(2\beta-2\theta)$
(II,S1) and (I,S2)	$-L(u-a)\sin(\beta)+L(v-b)\cos(\beta)$
(III,S1) and (IV,S2)	$-L(u-a)\sin(\beta-2\theta)-L(v-b)\cos(\beta-2\theta)-L^2t\ \sin(2\beta-2\theta)$
(IV,S1) and (III,S2)	$L(u-a)\sin(\beta)-L(v-b)\cos(\beta)$

Table 2
Identification of the quadrant

$I_1^{\mathrm{I}}=-I_1^{\mathrm{III}}$	$L(u-a)\cos(2\theta)+L(v-b)\sin(2\theta)+\frac{L^2}{2}\sin(2\theta)$
$I_1^{\mathrm{II}}=-I_1^{\mathrm{IV}}$	$-L(u-a)$
$I_2^{\mathrm{I}}=-I_2^{\mathrm{III}}$	$L(v-b)$
$I_2^{\mathrm{II}}=-I_2^{\mathrm{IV}}$	$-L(u-a)\sin(2\theta)+L(v-b)\cos(2\theta)-\frac{L^2}{2}\sin(2\theta)$

Table 3
Dependence of $I_1(u,v,L)$ and $I_2(u,v,L)$ on L

Quadrant	$I_1(u,v,L)$	$I_2(u,v,L)$
$x>0$ and $y>0$ (I)	Positively quadratic in L	Positively linear in L
$x<0$ and $y>0$ (II)	Positively linear in L	Negatively quadratic in L
$x<0$ and $y<0$ (III)	Negatively quadratic in L	Positively linear in L
$x>0$ and $y<0$ (IV)	Positively linear in L	Positively quadratic in L

We will see in Section 4.2, the estimation of (θ, a, b) is deduced from the identification of (11) and (12) through quadratic variations on various segments (cf. Theorem 3.1). However, as shown by Table 1, the behavior of the quadratic variations depends on the quadrant, where the random field $Y_{\theta,a,b}$ is observed. So, before estimating these parameters, we first have to identify the quadrant of $(O', \mathscr{X}, \mathscr{Y})$ where $Y_{\theta,a,b}$ is observed.

As θ is unknown, we are sure to be in situation (S1) only if $\beta=\pi/2$ and in situation (S2) only if $\beta=0$. Therefore, we restrict β to these two values for identification and/or estimation.

4.1. Identification of the quadrant

We use Table 1 and take $\beta=\pi/2$ and $\beta=0$. Set $I_1(u,v,L)=I_1(1;u,v,L,\pi/2)$, $I_2(u,v,L)=I_2(1;u,v,L,0)$ and I_i^Q for $I_i(u,v,L)$ computed in quadrant Q, $Q\in\{\mathrm{I, II, III, IV}\}$, $i=1,2$.

When $\theta\neq 0$, Table 3 allows us to identify the quadrant.

In Table 3, positively linear in L means that $I_i(u, v, L)$ is of the form $c \times L$ with $c > 0$, positively (respectively negatively) quadratic in L means that $I_i(u, v, L)$ is of the form $c' \times L + d \times L^2$ with $d > 0$ (respectively $d < 0$), $i = 1, 2$.

Let us give one example of identification assuming that $\theta \neq 0$. We first take $\beta = \pi/2$ and we use the fact that L is a parameter under our control. If $I_1(u, v, L)$ depends linearly on L, then the quadrant is II or IV. Secondly, we take $\beta = 0$: if $I_2(u, v, L)$ is positively quadratic in L the quadrant is IV, otherwise the limit $I_1(u, v, L)$ is negatively quadratic in L and the quadrant is II. Of course, this procedure is not unique, but all the procedures are consistent with each other.

We have just assumed that $\theta \neq 0$ but this is not a restriction. Indeed, we easily deduce from Table 2 that we can identify $\theta = 0$ as well as $\theta = \pi/4$.

Theorem 4.1.

- *$\theta = 0$ if and only if $I_1(u, v, L) + I_2(u, v, L)$ is positively linear in L.*
- *$\theta = \pi/4$ if and only if $I_1(u, v, L) - I_2(u, v, L)$ is purely quadratic in L.*

4.2. Estimation of the parameters

Once the quadrant is identified, we suppose, without any restriction, that $Y_{\theta,a,b}$ is observed in the positive quadrant I. We estimate (θ, a, b) by using a convenient set of quadratic variations that makes (θ, a, b) identifiable. However, there are different ways to estimate the parameters depending on the set of quadratic variations we choose that is depending on the choice of the set of segments $[A, B]$. At least we need a set of three quadratic variations. In fact, we decide to take four quadratic variations and we justify our choice in the sequel.

For the sake of simplicity we write $V_n(\beta, u, v, L)$ for $V_n(1; \beta, u, v, L)$ and we consider $V_n(0, u_0, v_1, L)$ and $V_n(\pi/2, u, v, L)$ for $(u, v) = (u_1, v_0), (u_1, v_1), (u_2, v_0)$, where u_0, u_1 and u_2 are distinct as well as v_0 and v_1. These four variations are taken along four distinct segments illustrated by Fig. 3.

We have almost surely, as n tends to infinity

$$V_n(0, u_0, v_1, L) \longrightarrow L(v_1 - b), \tag{13}$$

$$V_n\left(\frac{\pi}{2}, u, v, L\right) \longrightarrow L((u-a)\cos(2\theta) + (v-b)\sin(2\theta)) + \frac{L^2}{2}\sin(2\theta), \tag{14}$$

where $(u, v) = (u_1, v_0), (u_1, v_1), (u_2, v_0)$.

We deduce the following estimators:

$$\begin{aligned}\hat{a}_n = u_1 &+ \frac{(u_2 - u_1)}{(v_1 - v_0)(V_n\left(\frac{\pi}{2}, u_2, v_0, L\right) - V_n\left(\frac{\pi}{2}, u_1, v_0, L\right))} \\ &\times \left((v_0 - v_1)V_n\left(\frac{\pi}{2}, u_1, v_1, L\right)\right. \\ &\left. + \left(V_n(0, u_0, v_1, L) + \frac{L}{2}\right)\left(V_n\left(\frac{\pi}{2}, u_1, v_1, L\right) - V_n\left(\frac{\pi}{2}, u_1, v_0, L\right)\right)\right),\end{aligned} \tag{15}$$

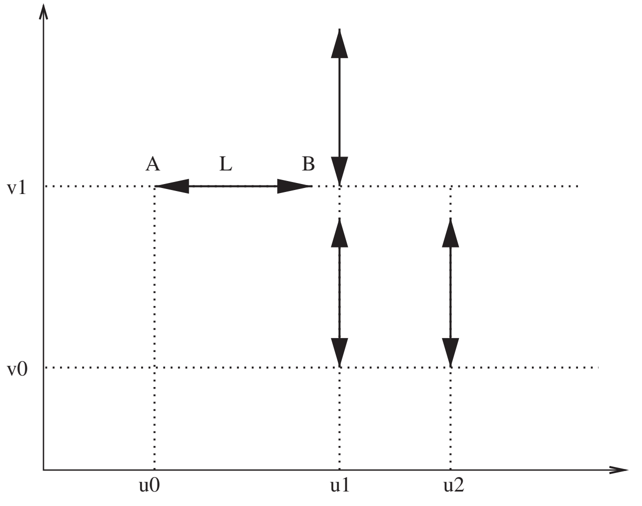

Fig. 3. Supports of the quadratic variations.

$$\hat{b}_n = v_1 - \frac{V_n(0, u_0, v_1, L)}{L}, \tag{16}$$

$$\hat{\theta}_n = \tfrac{1}{2} \arctan\left(\frac{(V_n(\frac{\pi}{2}, u_1, v_1, L) - V_n(\frac{\pi}{2}, u_1, v_0, L))(u_2 - u_1)}{(V_n(\frac{\pi}{2}, u_2, v_0, L) - V_n(\frac{\pi}{2}, u_1, v_0, L))(v_1 - v_0)}\right). \tag{17}$$

Theorem 4.2. *The estimators $\hat{a}_n$, $\hat{b}_n$ and $\hat{\theta}_n$ defined by* (15), (16) *and* (17) *converge almost surely to a, b and θ, as n tends to infinity.*

Proof. In this proof we mainly insist on the construction of the estimators, the convergence result follows easily from Theorem 3.1. Note that this proof can be viewed as an algorithm. First, we directly deduce $\hat{b}_n$ from (13). From (14), we get almost surely, as n tends to infinity

$$V_n\left(\frac{\pi}{2}, u_2, v_0, L\right) - V_n\left(\frac{\pi}{2}, u_1, v_0, L\right) \longrightarrow L(u_2 - u_1)\cos(2\theta),$$

so that we deduce the following estimator for $\cos(2\theta)$

$$\widehat{\cos(2\theta)}_n = \frac{V_n(1; \pi/2, u_2, v_0, L) - V_n(1; \pi/2, u_1, v_0, L)}{L(u_2 - u_1)}.$$

As this estimator does not necessarily belong to $[-1, 1]$, we take a fourth quadratic variation

$$V_n\left(\frac{\pi}{2}, u_1, v_1, L\right) - V_n\left(\frac{\pi}{2}, u_1, v_0, L\right) \longrightarrow L(v_1 - v_0)\sin(2\theta),$$

so that we deduce the following estimators for $\sin(2\theta)$ and θ

$$\widehat{\sin(2\theta)}_n = \frac{V_n(1;\pi/2,u_1,v_1,L) - V_n(1;\pi/2,u_1,v_0,L)}{L(v_1-v_0)} \quad \hat{\theta}_n = \tfrac{1}{2}\arctan\left(\frac{\widehat{\sin(2\theta)}_n}{\widehat{\cos(2\theta)}_n}\right).$$

Finally, we deduce from (14)

$$\hat{a}_n = u_1 + \frac{L(v_0-\hat{b}_n)\widehat{\sin(2\theta)}_n + \frac{L^2}{2}\widehat{\sin(2\theta)}_n - V_n(\frac{\pi}{2},u_1,v_0,L)}{L\,\widehat{\cos(2\theta)}_n}. \quad \square$$

Remark 4.1. To better use the available information contained in the open subset $\mathscr{B}$ where $Y_{\theta,a,b}$ is observed, we could consider N sets of four quadratic variations. From each set $j=1,\dots,N$, we get estimators $\hat{\theta}_n^j$, $\hat{a}_n^j$ and $\hat{b}_n^j$. Thus, we could propose a robust-like version of the estimators as the empirical means of these estimators.

5. An experimental study

Here, we consider that the transformation is the identity function, that is $(\theta,a,b)=(0,0,0)$. Moreover, we suppose that we observe $Y_{\theta,a,b}$ in the square $]0,2]\times]0,2]$.

We simulate a standard discrete Brownian sheet W (or equivalently $Y_{\theta,a,b}$ in this framework) on the regular square grid $[0,1/n,\dots,(2n-1)/n,2]\times[0,1/n,\dots,(2n-1)/n,2]$ with mesh $1/n=10^{-3}$ as follows:

1. we simulate $4n^2$ independent centered Gaussian variables with variance n^{-2}, denoted as $\varepsilon(k/n,l/n)$, $1\leqslant k,l\leqslant 2n$;
2. the values $w(i/n,j/n)$, $1\leqslant i,j\leqslant 2n$, of the Brownian sheet W are

$$w\left(\frac{i}{n},\frac{j}{n}\right) = \sum_{k=1}^{i}\sum_{l=1}^{j}\varepsilon\left(\frac{k}{n},\frac{l}{n}\right).$$

We repeat independently this simulation of a standard discrete Brownian sheet 5000 times.

First, from Theorem 4.1 we must check that $I_1(u,v,L)+I_2(u,v,L)$ (which is the almost sure limit of $V_n(\pi/2,u,v,L)+V_n(0,u,v,L)$ as $n\to\infty$) is positively linear in L. We choose $u=v=0.5$ so that from Table 2 we must get $I_1(u,v,L)+I_2(u,v,L)=L$. We take 6 different values for L, $L\in C=\{1,1.1,1.2,1.3,1.4,1.5\}$. Let $V_j(L)=V_n(\pi/2,u,v,L)+V_n(0,u,v,L)$ be the estimation of $I_1(u,v,L)+I_2(u,v,L)$ corresponding to the simulation j, $j=1,\dots,5000$. Fig. 4, which represents the (linear) empirical dependence of $V(L)$ on $L\in C$, is consistent with Theorem 4.1 which asserts the theoretical linear dependence of $V(L)$ on L. We can thus conclude that $\theta=0$.

Nevertheless, to see how the formulas of Section 4.2 can apply in this experimental study, we still propose to estimate θ in the sequel.

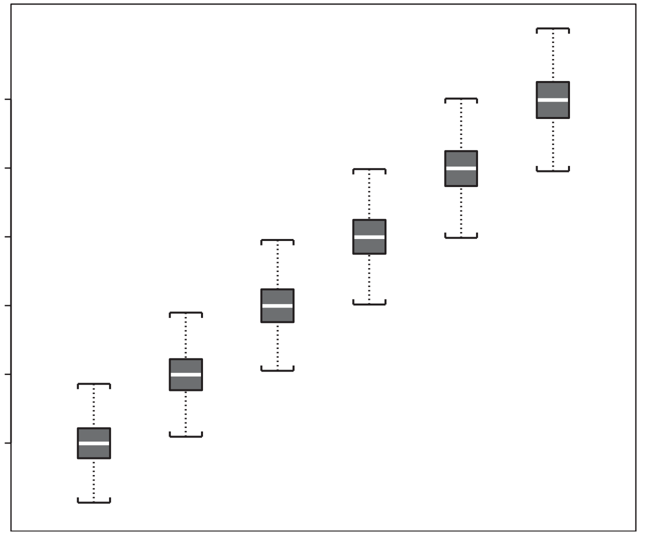

Fig. 4. Boxplots for $V(L)$ from the values $V_j(L)$, $j = 1, \dots, 5000$, with respect to $L \in \{1, 1.1, 1.2, 1.3, 1.4, 1.5\}$.

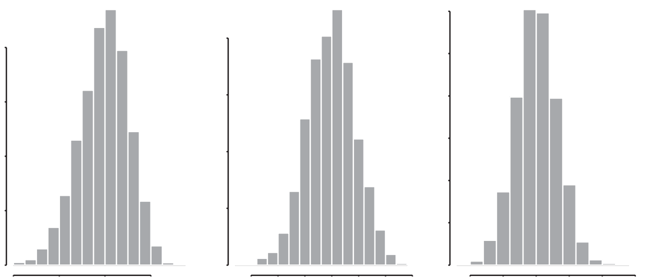

Fig. 5. Histograms of the estimations of a (left-hand plot), b (middle plot) and θ (right-hand plot).

The unknown parameters (θ, a, b) are then estimated with the help of the four quadratic variations introduced in Section 4.2, where we take $u_0 = v_0 = 0$, $u_1 = v_1 = 1$, $u_2 = 2$ and $L = 1$.

Finally, to give an evaluation of the quality of the estimators $(\hat{\theta}_n, \hat{a}_n, \hat{b}_n)$ for (θ, a, b), we compute the estimations of θ, a and b for each of the 5000 simulations. Fig. 5 represents the histogram of these 5000 estimations. The mean and the standard error computed over the 5000 estimations are indicated above the corresponding histogram. Note that 0 belongs well to the empirical 95% confidence interval for each parameter.

With respect to this simulation study, we conclude that using quadratic variations allows us to provide a satisfying estimation of the unknown canonical coordinate system of the standard Brownian sheet.

Otherwise, remark that from (16) and Theorem 4.1 in Perrin (1999), we obtain the asymptotic variance of $\hat{b}_n$. More precisely

$$\mathrm{Var}(\hat{b}_n) = \frac{1}{L^2}\mathrm{Var}(V_n(0,0,1,1)) = \frac{1}{nL^2}\int_0^1 2\alpha_2^2(w;0,0,1,1)\,\mathrm{d}w,$$

where $\alpha_2^2(w;0,0,1,1)$ is given by (10). Therefore, we get $\mathrm{Var}(\hat{b}_n) = 2/n = 0.002$, which is very comparable to the empirical variance of $\hat{b}_n$ equal to $0.04432^2 = 0.0019$. The asymptotic 95% confidence interval for b is $[-0.0877, 0.0877]$.

6. Discussion

We first point out three direct developments of the present work.

We do not give the asymptotic variances for $\hat{a}_n$ and $\hat{\theta}_n$ for the following reason: as defined by (15) and (17), $\hat{a}_n$ and $\hat{\theta}_n$ depend non-linearly on four quadratic variations corresponding to four non-linear segments unlike $\hat{b}_n$, which depends on just one quadratic variation. As far as we know, there is no result about the estimate of the asymptotic variance of a non-linear combination of disjoined quadratic variations. We are currently working on this point. Moreover, once we would have such a result it would be interesting to work on an optimal choice of a identifying family, according to a variance criterion for instance. More precisely, there are different ways to estimate the parameters depending on the set of quadratic variations we choose. The choice we have done in Section 4.2 is perhaps not the best one in terms of precision.

Concerning the identification of the quadrant, statistical properties of its estimation as well as an identification test would be useful as well.

One further direct extension concerning this work would be to consider a general C^1 transformation instead of an isometry. Call $\Phi = (\Phi_1, \Phi_2)$ this general transformation, the model would still be (1) but with the canonical coordinates given by $x(u,v) = \Phi_1(u,v)$ and $y(u,v) = \Phi_2(u,v)$. Due to the local and analytical nature of the method we introduce in this paper, it would be worth working on such a generalization.

Second, in Cohen et al. (2004), we generalize our results to the standard fractional Brownian sheet, that is a centered Gaussian field with covariance

$$E(W(x_1,y_1)W(x_2,y_2)) = \frac{1}{4}(|x_1|^{2H_1} + |x_2|^{2H_1} - |x_1 - x_2|^{2H_1}) \times (|y_1|^{2H_2} + |y_2|^{2H_2} - |y_1 - y_2|^{2H_2}),$$

where $(H_1, H_2) \in]0,1]^2$. Thus, the observed field Y depends on the chosen coordinate system as well as on the values of parameters (H_1, H_2), unlike the standard Brownian sheet which only depends on the coordinate system (for the standard Brownian sheet $H_1 = H_2 = \frac{1}{2}$).

The difficulty we are currently dealing with relies on the estimation of the parameters of the transformation (θ, a, b) together with the estimation of the parameters (H_1, H_2) of the field. However, the identification method we developed in the present paper can be applied to this more sophisticated problem. More precisely, quadratic variations can be useful for estimating both the coordinate system and the parameters of the random field itself. For this, we generalize in Cohen et al. (2004) the main result of Gladyshev (1961) to generalized quadratic variations.

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