Brownian penalisations

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By puting an adequate weight Γ_t on the Wiener measure W, constructed on $(\Omega = C(\mathbb{R}_+, \mathbb{R}), (X_t, \mathcal{F}_t)_{t\geq 0})$, where $X_s(\omega) = \omega(s)$, $\omega \in \Omega$, $s \geq 0$, denotes the canonical process, and, $(\mathcal{F}_t = \sigma\{X_s, s \leq t\}, t \geq 0)$ its natural filtration, we wish to show that :

$$W_t^\Gamma := \Gamma_t \cdot W_{|\mathcal{F}_t}$$

when restricted to $\mathcal{F}_s$, for any finite s, converges as $t \to \infty$, *i.e.* :

$$\forall s > 0, \forall F_s \in \mathcal{F}_s,\ W_s^\Gamma(F_s) \to W_\infty^\Gamma(F_s), \ \text{ as } t \to \infty.$$

Assuming that this holds, it is not difficult to show that W_∞^Γ induces a probability on $(\Omega, \mathcal{F}_\infty)$, such that :

$$\forall s > 0,\ (W_\infty^\Gamma)_{|\mathcal{F}_s} = M_s \cdot W_{|\mathcal{F}_s}$$

for certain martingale (M_s) with respect to $(W, (\mathcal{F}_s))$.

We then say that we have penalised W with the weight process $(\Gamma_t, t \geq 0)$. The process (X_t) under W_∞^Γ has a radically different behavior than under W.

We can look systematically for such of alterations of the Wiener measure, by taking weight processes involving the suppremum, or the local time in 0, or, in dimension 2, the winding process of planar Brownian motion, ...

The most natural penalisations of W are the so called Feynman-Kac penalisations :

$$\Gamma_t = {\exp\left(-\int_0^t q(X_s)ds\right)} \Big/ {E_W\left(\exp\left(-\int_0^t q(X_s)ds\right)\right)},$$

for some integrable function $q : \mathbb{R} \to \mathbb{R}_+$.

In these lectures we shall try to give a rough description of problems and answers connected to Brownian penalisations.