# Quadratic BSDEs with convex generators

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# Backward Stochastic Differential Equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s \qquad (\mathrm{E}_{\xi,f})$$

- $\xi$ is the terminal value : $\mathcal{F}_T$–measurable
- $f$ is the generator

- $(Y, Z)$ is the unknown
- $(Y, Z)$ has to be adapted to $\mathcal{F}$

**Pardoux–Peng, '90**

If $f$ is Lipschitz w.r.t. $(y, z)$ and

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(s,0,0)|^2\, ds\right] < +\infty$$

$(\mathrm{E}_{\xi,f})$ has a unique square integrable solution.

# Nonlinear Feynman-Kac's Formula

## Semilinear PDE (P)

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla_x u\sigma(t,x)) = 0, \quad u(T,.) = g,$$

$$\mathcal{L}u(t,x) = \frac{1}{2}\mathrm{trace}(\sigma\sigma^*\nabla_x^2 u(t,x)) + b(t,x)\cdot\nabla_x u(t,x).$$

## Linear part $\Longrightarrow$ SDE

$$X_t^{t_0,x_0} = x_0 + \int_{t_0}^t b\left(s, X_s^{t_0,x_0}\right)\, ds + \int_{t_0}^t \sigma\left(s, X_s^{t_0,x_0}\right)\, dB_s$$

## Nonlinear part $\Longrightarrow$ BSDE (B)

$$Y_t^{t_0,x_0} = g\left(X_T^{t_0,x_0}\right) + \int_t^T f\left(s, X_s^{t_0,x_0}, Y_s^{t_0,x_0}, Z_s^{t_0,x_0}\right)\, ds - \int_t^T Z_s^{t_0,x_0}\, dB_s$$

# Nonlinear Feynman-Kac's Formula

**If $u$ is smooth solution to (P)**

$$\left(u\left(t, X_t^{t_0,x_0}\right), \nabla_x u\sigma\left(t, X_t^{t_0,x_0}\right)\right) \quad \text{solves the BSDE (B)}$$

**Feynman-Kac's Formula**

$u(t,x) := Y_t^{t,x}$ is a (viscosity) solution to (P).

# Quadratic BSDEs

## A real valued BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T Z_s\, dB_s \qquad (\mathrm{E}_{\xi,f})$$

- $B$ is a Brownian motion in $\mathbf{R}^d$;
- $\xi$ is $\mathcal{F}_T$–measurable;
- the generator $f : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R}^d \longrightarrow \mathbf{R}$ is measurable and
    - $(y, z) \longmapsto f(t, y, z)$ is continuous
    - $f$ is quadratic with respect to $z$:

$$|f(t, y, z)| \leq \alpha(t) + \beta|y| + \frac{\gamma}{2}|z|^2$$

where $\beta \geq 0$, $\gamma > 0$ and $\alpha$ is a nonnegative process.

## The bounded case

If $\xi$ and $\alpha$ – or more generally $|\alpha|_1 := \int_0^T \alpha(s)\,ds$ – are bounded

- Existence
- Uniqueness, Comparison Theorem
- Stability

References:

- M. Kobylanski (1997 & 2000);
- J.-P. Lepeltier and J. San Martin (superlinear framework, 1998);
- M.-A. Morlais (non brownian setting, preprint)

**These results yield**

The Nonlinear Feynman-Kac Formula

# The unbounded case

- Boundedness of $\xi$ and $\alpha$ is not necessary to construct a solution;
- Exponential moment is enough !

**Theorem (Ph. B. & Ying HU)**

*Let* $\zeta := |\xi| + \int_0^T \alpha(s)\, ds$ *and let us assume that* $\mathbb{E}\left[\exp\left(\gamma e^{\beta T} \zeta\right)\right] < +\infty$.
*Then,* $(\mathrm{E}_{\xi,f})$ *has at least a solution such that*

$$|Y_t| \leq \frac{1}{\gamma} \log \mathbb{E}\left(\exp\left(\gamma e^{\beta T} \zeta\right) \,\middle|\, \mathcal{F}_t\right).$$

# Construction : $f \geq 0$, $\xi \geq 0$

### $(Y^n, Z^n)$ minimal solution

$$Y_t^n = \xi \wedge n + \int_t^T \mathbf{1}_{s \leq \sigma_n} f(s, Y_s^n, Z_s^n)\, ds - \int_t^T Z_s^n\, dB_s$$

$$\sigma_n = \inf \left\{ t \geq 0 : \int_0^t \alpha(s)\, ds \geq n \right\}$$

### Step 1: a priori estimate

$$0 \leq Y_t^n \leq \frac{1}{\gamma} \mathbb{E} \left( \exp \left[ \gamma e^{\beta T} \left( \xi + \int_0^T \alpha(s)\, ds \right) \right] \,\middle|\, \mathcal{F}_t \right)$$

### Step 2: taking the limit in $n$

Difficult step: **Localization procedure**

# The localization procedure

## Main Idea

Work on the interval $[0, \tau_k]$ where

$$\tau_k = \inf\left\{ t \geq 0 : \frac{1}{\gamma}\mathbb{E}\left(\exp\left[\gamma e^{\beta T}\left(\xi + \int_0^T \alpha(s)\, ds\right)\right] \,\middle|\, \mathcal{F}_t\right) \geq k \right\} \wedge T$$

## Set $Y_k^n(t) = Y_{t\wedge\tau_k}^n$, $Z_k^n(t) = \mathbf{1}_{t\leq\tau_k} Z_t^n$

$$Y_k^n(t) = Y_{\tau_k}^n + \int_{t\wedge\tau_k}^{\tau_k} \mathbf{1}_{s\leq\sigma_n} f\left(s,\, Y_k^n(s), Z_k^n(s)\right) ds - \int_{t\wedge\tau_k}^{\tau_k} Z_k^n(t)\, dB_s$$

## For fixed $k$

$$\left(Y_k^n\right)_{n\in\mathbf{N}} \text{ is nondecreasing}, \qquad 0 \leq Y_k^n(t) \leq k$$

## The localization procedure

### $k$ fixed, $\lim_{n \longrightarrow +\infty}$

$$Y_k(t) = \xi_k + \int_{t\wedge\tau_k}^{\tau_k} f(s, Y_k(s), Z_k(s))\, ds - \int_{t\wedge\tau_k}^{\tau_k} Z_k(s)\, dB_s, \quad \xi_k = \sup_{n\geq 1} Y^n_{\tau_k}$$

- By construction

$$Y_k(t) = Y_{k+1}(t \wedge \tau_k), \qquad Z_k(t) = \mathbf{1}_{t\leq\tau_k} Z_{k+1}(t)$$

- Define $(Y, Z)$ by

$$Y_t = Y_k(t), \qquad Z_t = Z_k(t) \qquad \text{if } t \leq \tau_k$$

$$Y_t = \xi_k + \int_{t\wedge\tau_k}^{\tau_k} f(s, Y_s, Z_s)\, ds - \int_{t\wedge\tau_k}^{\tau_k} Z_s\, dB_s$$

- $k \longrightarrow +\infty$ gives a solution

# Remarks

- superlinear framework and $\xi \in \mathrm{L}^1$: Ph. B. & Y. Hu
- Monotone continuous generators: Ph. B., J.-P. Lepeltier & J. San Martin
- Reflected Quadratic BSDEs: J.-P. Lepeltier & M. Xu
- Stochastic control : M. Fuhrman, Y. Hu & G. Tessitore

## Questions

Uniqueness ? Stability ? Feynman-Kac's formula ?

## Answers

When $f$ is convex (or concave) w.r.t. $z$

# Motivation: Stochastic Control Problem

## Controlled diffusion process

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)[dW_s + r(u_s)\, ds]$$

where $u$ takes its values in a nonempty closed set $C$.

## Minimize the cost functional

$$J(u) = \mathbb{E}\left[g(X_T) + \int_0^T G(t, X_t, u_t)\, dt\right]$$

over all the admissible controls $u$.

# Motivation

## Associated BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Z_s)\, ds - \int_t^T Z_s\, dB_s$$

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$$

$$f(t, x, z) = \inf \{ G(t, x, u) + r(u)z : u \in C \}$$

## Important feature of the generator

$z \longmapsto f(t, x, z)$ is **concave**

# Assumptions (H)

There exist $\beta \geq 0$, $\gamma \geq 0$ and a nonnegative process $\alpha$ s.t. $\mathbb{P}$–a.s.

- $f$ is Lipschitz w.r.t. $y$: for any $t, z$,

$$|f(t,y,z) - f(t,y',z)| \leq \beta\,|y - y'|$$

- quadratic growth in $z$:

$$|f(t,y,z)| \leq \alpha(t) + \beta|y| + \frac{\gamma}{2}|z|^2$$

- for any $t$, $y$, $z \longmapsto f(t,y,z)$ is a convex function;
- $\xi$ is $\mathcal{F}_T$–measurable and

$$\forall \lambda > 0, \qquad \mathbb{E}\left[\exp\left(\lambda\left[|\xi| + \int_0^T \alpha(s)\,ds\right]\right)\right] < +\infty.$$

# Some estimates

**Proposition**

$(\mathrm{E}_{\xi,f})$ *has a solution* $(Y,Z)$ *s.t.*

$$\forall p \geq 1, \qquad \mathbb{E}\left[\sup_{t\in[0,T]} e^{p|Y_t|} + \left(\int_0^T |Z_s|^2\, ds\right)^{p/2}\right] \leq C$$

*where* $C$ *depends only on* $p$, $T$ *and the exponential moments of* $|\xi| + |\alpha|_1$.

- The estimate for $Y$ comes directly from

$$|Y_t| \leq \frac{1}{\gamma} \log \mathbb{E}\left(\exp\left(\gamma e^{\beta T}(|\xi| + |\alpha|_1)\right) \,\middle|\, \mathcal{F}_t\right)$$

- For $Z$, standard computation starting from Itô's formula to

$$\frac{1}{\gamma^2}\left(e^{\gamma|Y_t|} - 1 - \gamma|Y_t|\right)$$

# Comparison theorem

### Theorem

*Let $(Y, Z)$ and $(Y', Z')$ be solution to $(\mathrm{E}_{\xi,f})$ and $(\mathrm{E}_{\xi',f'})$ where $(\xi, f)$ satisfies (H) and $Y$, $Y'$ belongs to $\mathcal{E}$ ($\mathcal{E}$ := exponential moment of all order).*
*If $\xi \leq \xi'$ and $f \leq f'$ then*

$$\forall t \in [0, T], \qquad Y_t \leq Y'_t$$

*If moreover, $Y_t = Y'_t$ then*

$$\mathbb{P}\left(\xi = \xi', \int_t^T f(s, Y'_s, Z'_s)\, ds = \int_t^T f'(s, Y'_s, Z'_s)\, ds\right) > 0.$$

*In particular, $(\mathrm{E}_{\xi,f})$ has a unique solution in the class $\mathcal{E}$.*

### Main idea

Estimate of $Y_t - \mu Y'_t$ for $\mu \in (0, 1)$.

# Proof: $f$ independent of $y$

Set, for $\mu \in (0,1)$, $U_t = Y_t - \mu Y'_t$, $V_t = Z_t - \mu Z'_t$.

$$U_t = U_T + \int_t^T F_s\,ds - \int_t^T V_s\,dB_s, \qquad F_s = f(s, Z_s) - \mu f'(s, Z'_s)$$

$$F_t = [f(t, Z_t) - \mu f(t, Z'_t)] + \mu\,[f(t, Z'_t) - f'(t, Z'_t)]$$

and $\delta f(t) := f(t, Z'_t) - f'(t, Z'_t) \leq 0$.

$$Z_t = \mu Z'_t + (1-\mu)\frac{Z_t - \mu Z'_t}{1-\mu}$$

$$\begin{aligned} f(t, Z_t) &= f\left(t, \mu Z'_t + (1-\mu)\frac{Z_t - \mu Z'_t}{1-\mu}\right) \\ \textbf{Convexity} \quad &\leq \mu f(t, Z'_t) + (1-\mu) f\left(t, \frac{Z_t - \mu Z'_t}{1-\mu}\right) \end{aligned}$$

$$f(t, Z_t) - \mu f(t, Z'_t) \leq (1-\mu) f\left(t, \frac{V_t}{1-\mu}\right) \leq (1-\mu)\alpha(t) + \frac{\gamma}{2(1-\mu)} |V_t|^2$$

$$F_t \leq \mu\delta f(t) + (1-\mu)\alpha(t) + \frac{\gamma}{2(1-\mu)} |V_t|^2$$

**Second step**

An exponential change of variable to remove the quadratic term

$$P_t = e^{cU_t}, \qquad Q_t = cP_t V_t, \qquad c \geq 0$$

$$P_t = P_T + c\int_t^T P_s \left(F_s - \frac{c}{2}|V_s|^2\right) ds - \int_t^T Q_s \, dB_s$$

$c = \frac{\gamma}{1-\mu}$ yield

$$P_t \leq P_T + \gamma \int_t^T \left(\alpha(s) + (1-\mu)^{-1}\mu\delta f(s)\right) P_s \, ds - \int_t^T Q_s \, dB_s$$

$$P_t \leq \mathbb{E}\left(\exp\left[\gamma\int_t^T \left(\alpha(s)+(1-\mu)^{-1}\mu\delta f(s)\right)\,ds\right]\,P_T\,\middle|\,\mathcal{F}_t\right)$$

$$P_T = \exp\left(\frac{\gamma}{1-\mu}(\xi-\mu\xi')\right) = \exp\left(\gamma\left(\xi+\frac{\mu}{1-\mu}\delta\xi\right)\right)$$

$$P_t \leq \mathbb{E}\left(\exp\left[\gamma\left(\xi+\int_0^T \alpha(s)\,ds\right)+\gamma\frac{\mu}{1-\mu}\left(\delta\xi+\int_t^T \delta f(s)\,ds\right)\right]\,\middle|\,\mathcal{F}_t\right)$$

In particular,

$$Y_t-\mu Y_t' \leq \frac{1-\mu}{\gamma}\log\mathbb{E}\left(\exp\left[\gamma\left(\xi+\int_0^T \alpha(s)\,ds\right)\right]\,\middle|\,\mathcal{F}_t\right)$$

and sending $\mu$ to 1, we get

$$Y_t - Y_t' \leq 0.$$

# Strict Comparison

If $Y_t = Y'_t$, then $P_t = e^{\gamma Y_t}$ and

$$0 < \mathbb{E}[P_t] \leq \mathbb{E}\left[\exp\left(\gamma\left(\xi + \int_0^T \alpha(s)\,ds\right) + \gamma\frac{\mu}{1-\mu}\left(\delta\xi + \int_t^T \delta f(s)\,ds\right)\right]\right)$$

Sending $\mu$ to 1,

$$0 < \mathbb{E}\left[\exp\left(\gamma\left[\xi + \int_0^T \alpha(s)\,ds\right]\right)\mathbf{1}_{\delta\xi + \int_t^T \delta f(s)\,ds = 0}\right]$$

▸ Press if late

# The general case

$$F_t = f(t, Y_t, Z_t) - \mu f'(t, Y'_t, Z'_t) = f(t, Y_t, Z_t) - \mu f(t, Y'_t, Z'_t) + \mu \delta f(t)$$

$$\begin{aligned} & f(t, Y_t, Z_t) - \mu f\left(t, Y'_t, Z'_t\right) \\ & \quad = \quad f(t, Y_t, Z_t) - \mu f\left(t, Y_t, Z'_t\right) + \mu\left(f\left(t, Y_t, Z'_t\right) - f\left(t, Y'_t, Z'_t\right)\right). \end{aligned}$$

Convexity

$$f(t, Y_t, Z_t) - \mu f\left(t, Y_t, Z'_t\right) \leq (1-\mu)(\alpha(t) + \beta |Y_t|) + \frac{\gamma}{2(1-\mu)} |V_t|^2$$

Linearization: $a(t) = \left(Y_t - Y'_t\right)^{-1}\left(f\left(t, Y_t, Z'_t\right) - f\left(t, Y'_t, Z'_t\right)\right) \mathbf{1}_{Y_t - Y'_t \neq 0}$

$$\mu\left(f\left(t, Y_t, Z'_t\right) - f\left(t, Y'_t, Z'_t\right)\right) = \mu a(t)\left(Y_t - Y'_t\right) \leq a(t) U_t + (1-\mu)\beta |Y_t|$$

$$F_t \leq \mu \delta f(t) + (1-\mu)(\alpha(t) + 2\beta |Y_t|) + \frac{\gamma}{2(1-\mu)} |V_t|^2 + a(t) U_t.$$

# The general case

Set $E_t = \exp\left(\int_0^t a(s)\, ds\right)$, $\widetilde{U}_t = E_t\, U_t$ and $\widetilde{V}_t = E_t\, V_t$. Then,

$$\widetilde{U}_t = \widetilde{U}_T + \int_t^T \widetilde{F}_s\, ds - \int_t^T \widetilde{V}_s\, dB_s$$

with, since $|a(t)| \le \beta$,

$$\widetilde{F}_t \le \mu E_t\, \delta f(t) + (1-\mu) E_t(\alpha(t) + 2\beta |\, Y_t|) + \frac{\gamma\, e^{\beta T}}{2(1-\mu)} \left| \widetilde{V}_t \right|^2$$

This is the same inequality as before.

# Stability

Assume that $(\xi_n, f_n)$ satisfies (H) with $\alpha_n$, $\beta$, $\gamma$ and

$$\forall\lambda > 0, \qquad \sup_{n\geq 1} \mathbb{E}\left[\exp\left\{\lambda\left(|\xi_n| + |\alpha_n|_1\right)\right\}\right] < +\infty.$$

## Theorem

*If* $\xi_n \longrightarrow \xi$ $\mathbb{P}$*–p.s. and* $dt \otimes d\mathbb{P}$*–a.e.,* $\forall(y,z)$, $f_n(t,y,z) \longrightarrow f(t,y,z)$, *then*

$$\forall p \geq 1, \qquad \mathbb{E}\left[\exp\left(p \sup_{t\in[0,T]} |Y_t - Y_t^n|\right) + \left(\int_0^T |Z_s - Z_s^n|^2\, ds\right)^{p/2}\right] \longrightarrow 0.$$

## Proof.

Same method as in the proof of comparison theorem to

$$Y_t - \mu Y_t^n, \qquad Y_t^n - \mu Y_t$$

$\square$

# Application to PDEs

- Probabilistic representation for

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla_x u\sigma(t,x)) = 0, \quad u(T,.) = g,$$

$$\mathcal{L}u(t,x) = \frac{1}{2}\mathrm{trace}(\sigma\sigma^*\nabla_x^2 u(t,x)) + b(t,x)\cdot\nabla_x u(t,x).$$

- The SDE: $X^{t_0,x_0}$ solution to

$$X_t = x_0 + \int_{t_0}^t b(s,X_s)\,ds + \int_{t_0}^t \sigma(s,X_s)\,dB_s$$

- The BSDE: $(Y^{t_0,x_0}, Z^{t_0,x_0})$ solution to

$$Y_t = g\left(X_T^{t_0,x_0}\right) + \int_t^T f\left(s, X_s^{t_0,x_0}, Y_s, Z_s\right)\,ds - \int_t^T Z_s\,dB_s$$

- Nonlinear Feynman-Kac's formula: $u(t,x) := Y_t^{t,x}$ is a viscosity solution

# Assumptions

- $b$, $\sigma$, $f$ and $g$ are continuous;
- $b$, $\sigma$ Lipschitz w.r.t. $x$

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq \beta|x - x'|;$$

- restriction: $\sigma$ is bounded;
- $f$ is Lipschitz w.r.t. $y$

$$|f(t,x,y,z) - f(t,x,y',z)| \leq \beta|y - y'|;$$

- $z \longmapsto f(t,x,y,z)$ is convex;
- $\exists p < 2$ s.t.

$$|g(x)| + |f(t,x,y,z)| \leq C\left(1 + |x|^p + |y| + |z|^2\right).$$

# First properties

## Proposition

*$u(t,x) := Y_t^{t,x}$ is continuous and*

$$|u(t,x)| \leq C\left(1+|x|^p\right).$$

## Proof.

- Since $\sigma$ is bounded

$$\forall \lambda > 0, \qquad \mathbb{E}\left[\exp\left(\lambda \sup_{t\in[0,T]} |X_t^{t_0,x_0}|^p\right)\right] \leq e^{C(1+|x|^p)}$$

- a priori estimate Here

$$|u(t,x)| \leq C\left(1+|x|^p\right)$$

- Stability $\Longrightarrow$ Continuity

□

# $u$ is a viscosity solution

## Definition

A continuous function $u$ s.t. $u(T, \cdot) = g$ is a viscosity subsolution (**supersolution**) if, whenever $u - \varphi$ has a local maximum (**minimum**) at $(t_0, x_0)$ where $\varphi$ is $\mathcal{C}^{1,2}$,

$$\partial_t \varphi(t_0, x_0) + \mathcal{L}\varphi(t_0, x_0) + f(t_0, x_0, u(t_0, x_0), \nabla_x u\sigma(t_0, x_0)) \geq 0, \quad (\leq 0)$$

**Solution = Subsolution + Supersolution**

## Proposition

$u(t, x) := Y_x^{t,x}$ *is a viscosity solution to the PDE.*

## Proof.

- Markov property : $u(t, X_t^{t_0,x_0}) = Y_t^{t_0,x_0}$
- Comparison theorem

□

# To do or at least try to do

- Replace the Lipschitz continuity in $y$ by monotony
- Weaken the integrability assumptions

$$|g(x)| + |f(t, x, y, z)| \leq C \left(1 + |x|^2 + |y| + |z|^2\right)$$

- Prove uniqueness and stability without convexity

$$|f(t, y, z) - f(t, y, z')| \leq C\, |z - z'|\, (1 + |z| + |z'|)$$

- Conclude the talk $\boxed{\surd}$