Symplectic reduction for finite-dimensional Hamiltonian systems

Marine Fontaine

School of Mathematics
The University of Manchester

Université de Rennes 1, 23 April 2014

Symplectic manifold

A symplectic manifold (P, ω) is a pair consisting of a smooth manifold P together with a differential 2-form $\omega \in \Omega^2(P)$ which is

- closed, that is $\mathrm{d}\omega = 0$
- non-degenerate

The non-degeneracy condition implies that $\forall z \in P$, there is an injection $\omega(z)^\flat : T_z P \to T_z^* P$ defined by

$$\omega(z)^\flat(v_z)(w_z) := \omega(z)(v_z, w_z)$$

for all $v_z, w_z \in T_z P$. If it is an isomorphism, we say that ω is strongly non-degenerate.

Examples of symplectic manifolds

Example 1. The product $P = V \times V^*$ where V is a finite dimensional real vector space with the constant 2-form $\omega\left((v,\alpha),(w,\beta)\right) = \langle \alpha, w\rangle - \langle \beta, v\rangle$ is a symplectic vector space.

Example 2. The cylinder $S^1 \times \mathbb{R}$ with coordinates (θ, p) is a symplectic manifold with $\omega = d\theta \wedge dp$.

Example 3. The torus $\mathbb{T}^2$ with periodic coordinates (θ, ϕ) is a symplectic manifold with $\omega = d\theta \wedge d\phi$.

Example 4. The two-sphere S^2 of radius r is a symplectic manifold with the standard area element $\omega = r^2 \sin(\theta) d\theta \wedge d\phi$.

Hamiltonian systems on symplectic manifolds

Let (P, ω) be a symplectic manifold. A vector field $X \in \mathfrak{X}(P)$ is a Hamiltonian vector field if there is a differentiable function $H : P \to \mathbb{R}$ such that, for all $z \in P$ and $v_z \in T_zP$,

$$\omega(z)(X(z), v_z) = DH(z) \cdot v_z.$$

In that case, we write $X =: X_H$.

We also define the associated Hamiltonian dynamical system, whose points z evolve in time, by the differential equation

$$\dot{z} = X_H(z).$$

Darboux' Theorem. Let (P, ω) be a strong symplectic manifold. Then, in a neighborhood of each $z \in P$, there is a local coordinate chart in which ω is constant.

Corollary. If (P, ω) is a finite-dimensional symplectic manifold, then P is even dimensional and, in a neighborhood of $z \in P$, there are local coordinates $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ such that

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

In such coordinates $(q^1, \cdots, q^n, p_1, \cdots, p_n)$, the relation

$$\omega(z)(X_H(z), v_z) = DH(z) \cdot v_z = \langle grad_z(H), v_z \rangle.$$

implies that $X_H(z) = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i})$ and then, the associated dynamical system is

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \end{aligned}$$

Example: the Heavy Top

Two identifications of $TSO(3)$ with the product $SO(3) \times \mathbb{R}^3$:

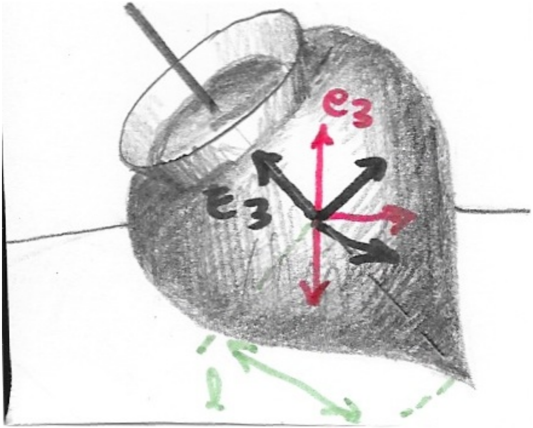
- The body variables $(\Lambda, \delta\boldsymbol{\Theta}) \in SO(3) \times \mathbb{R}^3$ corresponds to the left trivialization of $TSO(3)$.
- The space variables $(\Lambda, \delta\theta) \in SO(3) \times \mathbb{R}^3$ corresponds to the right trivialization of $TSO(3)$.

A pair $(\Lambda, \delta\boldsymbol{\Lambda})$ is an element of $TSO(3)$ if and only if $\Lambda \in SO(3)$ and

$$\Lambda\widehat{\delta\boldsymbol{\Theta}} = \delta\boldsymbol{\Lambda} = \widehat{\delta\theta}\Lambda$$

for some vectors $\delta\boldsymbol{\Theta}$ and $\delta\theta$ in $\mathbb{R}^3$. The isomorphism $\hat{\ }: \mathbb{R}^3 \to \mathfrak{so}(3)$ identifies a vector $\mathbf{w} \in \mathbb{R}^3$ with a skewsymmetric matrix $\widehat{\mathbf{w}} \in \mathfrak{so}(3)$ satisfying $\widehat{\mathbf{w}}\mathbf{x} = \mathbf{w} \times \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Example: the Heavy Top



The phase space $T^*SO(3)$ can be endowed with a symplectic structure.

Example: the Heavy Top

Two identifications of $T^*SO(3)$ with the product $SO(3) \times \mathbb{R}^3$:

- The body variables $(\Lambda, \mathbf{\Pi}) \in SO(3) \times \mathbb{R}^3$ corresponds to the left trivialization of $T^*SO(3)$.
- The space variables $(\Lambda, \pi) \in SO(3) \times \mathbb{R}^3$ corresponds to the right trivialization of $T^*SO(3)$.

A pair $(\Lambda, \mathbf{\Pi}_\Lambda)$ is an element of $T^*SO(3)$ if and only if $\Lambda \in SO(3)$ and

$$\Lambda\breve{\mathbf{\Pi}} = \mathbf{\Pi}_\Lambda = \breve{\pi}\Lambda$$

for some vectors $\mathbf{\Pi}$ and π in $\mathbb{R}^3$. The isomorphism $\breve{\ } : \mathbb{R}^3 \to \mathfrak{so}(3)^*$ identifies a vector $\mathbf{v} \in \mathbb{R}^3$ with the covector $\breve{\mathbf{v}} \in \mathfrak{so}(3)^*$ uniquely defined by $\langle \breve{\mathbf{v}}, \widehat{\mathbf{w}} \rangle = \mathbf{v} \cdot \mathbf{w}$ for all $\widehat{\mathbf{w}} \in \mathfrak{so}(3)$.

Example: the Heavy Top

For details see: *The heavy top: a geometric treatment* by D. Lewis, T. Ratiu, J.C. Simo and J.E. Marsden 1991

For $(\Lambda, \mathbf{\Pi}) \in SO(3) \times \mathbb{R}^3$, the Hamiltonian in body coordinates is

$$\boxed{H(\Lambda, \mathbf{\Pi}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{\Omega} + mg\ell\Lambda^T \mathbf{e_3} \cdot \mathbf{e_3}}$$

where $\mathbf{\Omega} = \mathbb{I}_0^{-1}\mathbf{\Pi}$ is the body angular velocity of the top and $\mathbb{I}_0$ is the reference inertia tensor. The corresponding Hamiltonian vector field

$$X_H(\Lambda, \mathbf{\Pi}) = (\Lambda\widehat{\mathbf{X}}_\Theta, \mathbf{X}_\Pi) \in T_\Lambda SO(3) \times \mathbb{R}^3$$

must satisfy

$$\boxed{\omega(\Lambda, \mathbf{\Pi})\Big((\Lambda\widehat{\mathbf{X}}_\Theta, \mathbf{X}_\Pi), (\Lambda\widehat{\delta\mathbf{\Theta}}, \delta\mathbf{\Pi})\Big) = DH(\Lambda, \mathbf{\Pi}) \cdot (\Lambda\widehat{\delta\mathbf{\Theta}}, \delta\mathbf{\Pi}).}$$

For all $(\Lambda\widehat{\delta\mathbf{\Theta}}, \delta\mathbf{\Pi}) \in T_\Lambda SO(3) \times \mathbb{R}^3$.

Example: the Heavy Top

The symplectic form on $SO(3) \times \mathbb{R}^3$ is given by

$$\boxed{\omega(\Lambda, \mathbf{\Pi})\left((\Lambda\widehat{\mathbf{X}}_\Theta, \mathbf{X}_\Pi), (\Lambda\widehat{\delta\mathbf{\Theta}}, \delta\mathbf{\Pi})\right) = (\mathbf{\Pi} \times \mathbf{X}_\Theta - \mathbf{X}_\Pi) \cdot \delta\mathbf{\Theta} + \mathbf{X}_\Theta \cdot \delta\mathbf{\Pi}}$$

while the directional derivative of the Hamiltonian H in the direction $(\Lambda\widehat{\delta\mathbf{\Theta}}, \delta\mathbf{\Pi}) \in T_\Lambda SO(3) \times \mathbb{R}^3$ is

$$\boxed{DH(\Lambda, \mathbf{\Pi}) \cdot (\Lambda\widehat{\delta\mathbf{\Theta}}, \delta\mathbf{\Pi}) = -mg\ell(\Lambda^T\mathbf{e_3} \times \mathbf{e_3}) \cdot \delta\mathbf{\Theta} + \mathbf{\Omega} \cdot \delta\mathbf{\Pi}.}$$

We find eventually the Hamiltonian vector field

$$X_H(\Lambda, \mathbf{\Pi}) = \left(\Lambda\widehat{\mathbf{X}}_\Theta, \mathbf{X}_\Pi\right) = \left(\Lambda\widehat{\mathbf{\Omega}}, \mathbf{\Pi} \times \mathbf{\Omega} + mg\ell(\Lambda^T\mathbf{e_3} \times \mathbf{e_3})\right).$$

Example: the Heavy Top

Since

$$(\dot{\Lambda}, \dot{\mathbf{\Pi}}) = X_H(\Lambda, \mathbf{\Pi}),$$

the associated Hamiltonian dynamical system (in body coordinates) is

$$\begin{cases} \dot{\Lambda} & = \Lambda\widehat{\mathbf{\Omega}} \\ \dot{\mathbf{\Pi}} & = \mathbf{\Pi} \times \mathbf{\Omega} + mg\ell(\Lambda^T \mathbf{e_3} \times \mathbf{e_3}) \end{cases}$$

where $\mathbf{\Omega} = \mathbb{I}_0^{-1}\mathbf{\Pi} \in \mathbb{R}^3$ corresponds to the angular velocity in the body frame and $\Lambda \in SO(3)$.

Infinitesimal generators

Consider a smooth action of a Lie group G on a symplectic manifold (P, ω)

$$\begin{aligned} \phi : G \times P &\to P \\ (g, z) &\mapsto \phi_g(z) \end{aligned}$$

The infinitesimal generator of this action corresponding to a Lie algebra element $\xi \in \mathfrak{g}$ is the vector field $\xi_P \in \mathfrak{X}(P)$ given by

$$\xi_P(z) := \left.\frac{\mathsf{d}}{\mathsf{dt}}\right|_{t=0} \phi_{exp(t\xi)}(z).$$

We are interested in the case in which the vector field ξ_P is globally Hamiltonian, that is, when we have $\forall z \in P$ and $v_z \in T_z P$,

$$\omega(z)(\xi_P(z), v_z) = dj_\xi(z) \cdot v_z$$

for some smooth function $j_\xi : P \to \mathbb{R}$.

Momentum map

Let a Lie group G acting on the symplectic manifold (P, ω). If the action does preserve the symplectic form and, for all $\xi \in \mathfrak{g}$, the vector field ξ_P is globally Hamiltonian, the map $\mathbf{J} : P \to \mathfrak{g}^*$ defined by

$$\langle \mathbf{J}(z), \xi \rangle = j_\xi(z)$$

for all $\xi \in \mathfrak{g}$ and $z \in P$ is called the momentum map of the action.

Continuous symmetry

A continuous symmetry of a Hamiltonian system (P, ω, H) is a smooth vector field $X \in \mathfrak{X}(P)$ that

- preserves the Hamiltonian function, $\pounds_X H = 0$
- respects the structure of the state-space, $\pounds_X \omega = 0$

Example: the Heavy Top

The isotropy group $SO(3)_{\mathbf{e_3}} = \{B \in SO(3) \mid B\mathbf{e_3} = \mathbf{e_3}\} \simeq S^1$ acts on $SO(3) \times \mathbb{R}^3$ (left action) by

$$\phi_B(\Lambda, \mathbf{\Pi}) = (B\Lambda, \mathbf{\Pi}).$$

- This action corresponds to a rotation of the top about the vertical axis $\mathbf{e_3}$.
- This action lets the Hamiltonian $H(\Lambda, \mathbf{\Pi}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{\Omega} + mg\ell\Lambda^T\mathbf{e_3} \cdot \mathbf{e_3}$ invariant.
- Can be shown that this action preserves the symplectic form, that is, $\phi_B^*\omega = \omega$ for all $B \in SO(3)_{\mathbf{e_3}}$.

The infinitesimal generators are continuous symmetries of the Hamiltonian system $(SO(3) \times \mathbb{R}^3, \omega, H)$.

Hamiltonian version of Noether's theorem

Noether's Theorem. Assume that the Hamiltonian G-system (P, ω, H, G) admits a momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. Then, if every infinitesimal generator $\xi_P \in \mathfrak{X}(P)$ of the action of G is a continuous symmetry, the momentum map is a constant of motion, that is,

$$\mathbf{J} \circ \phi_t = \mathbf{J}$$

where ϕ_t is the flow of the Hamiltonian vector field X_H.

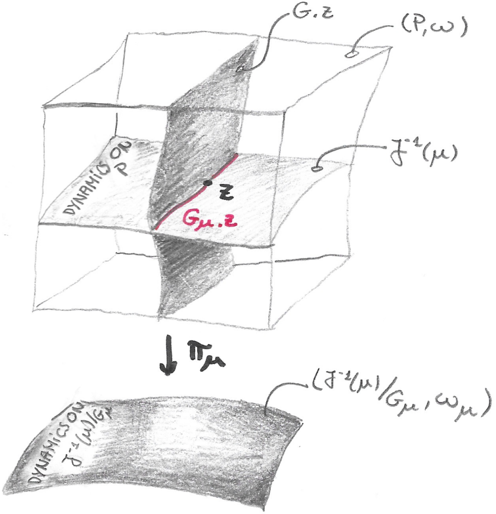
Symplectic reduction theorem

Reduction of symplectic manifolds with symmetry by J. Marsden and A. Weinstein 1974

Symplectic reduction theorem. Let (P, ω) be a symplectic manifold. Assume there is a smooth, free and proper action of a Lie group G on P, which preserves the symplectic form, together with an (equivariant) momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. For $\mu \in \mathfrak{g}^*$ a regular value of $\mathbf{J}$, the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form ω_μ uniquely characterized by the relation $\pi_\mu^* \omega_\mu = i_\mu^* \omega$ where

- $i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow P$ is the natural inclusion
- $\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ is the quotient map
- $G_\mu := \{g \in G \mid Ad_g^* \mu = \mu\}$ the isotropy group at μ for the coadjoint action on $\mathfrak{g}^*$

Symplectic reduction theorem



Example: the Heavy Top

The momentum map $\mathbf{J} : SO(3) \times \mathbb{R}^3 \to \mathbb{R}$ corresponding to this action is

$$\mathbf{J}(\Lambda, \mathbf{\Pi}) = \mathbf{\Pi} \cdot \Lambda^T \mathbf{e_3}$$

A level set $\mathbf{J}^{-1}(\mu)$ corresponds to the conserved quantity $\mathbf{\Pi} \cdot \Lambda^T \mathbf{e_3} = \mu$.

- $\mathbf{J}^{-1}(\mu)$ is a smooth manifold.
- The induced S^1-action preserves $\mathbf{J}^{-1}(\mu)$ and is **smooth**, **free** and **proper**.
- The reduced space $\mathbf{J}^{-1}(\mu)/S^1$ is a smooth manifold.
- $(\mathbf{J}^{-1}(\mu)/S^1, \omega_\mu)$ is symplectomorphic to $(\mathcal{P}_\mu, \omega_\mu^{\mathcal{P}})$ where $\mathcal{P}_\mu = \left\{(\mathbf{\Gamma}, \mathbf{\Pi}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \ \| \mathbf{\Gamma} \| = 1 \text{ et } \mathbf{\Pi} \cdot \mathbf{\Gamma} = \mu\right\}$ and $\omega_\mu^{\mathcal{P}}$ will be defined below.

Example: the Heavy Top

Taking $(\mathcal{P}_\mu, \omega_\mu^{\mathcal{P}})$ as a model space for the reduced space $(\mathbf{J}^{-1}(\mu)/S^1, \omega_\mu)$, the reduced Hamiltonian is

$$\boxed{H_\mu(\mathbf{\Gamma}, \mathbf{\Pi}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{\Omega} + mg\ell \mathbf{\Gamma} \cdot \mathbf{e_3}}$$

for $(\mathbf{\Gamma}, \mathbf{\Pi}) \in \mathcal{P}_\mu$. The corresponding reduced Hamiltonian vector field

$$X_{H_\mu}(\mathbf{\Gamma}, \mathbf{\Pi}) = (\mathbf{\Gamma} \times \mathbf{X}_\Theta, \mathbf{X}_\Pi) \in T_{(\mathbf{\Gamma}, \mathbf{\Pi})}\mathcal{P}_\mu$$

must satisfy

$$\boxed{\omega_\mu^{\mathcal{P}}(\mathbf{\Gamma}, \mathbf{\Pi})\left((\mathbf{\Gamma} \times \mathbf{X}_\Theta, \mathbf{X}_\Pi), (\mathbf{\Gamma} \times \delta\mathbf{\Theta}, \delta\mathbf{\Pi})\right) = DH_\mu(\mathbf{\Gamma}, \mathbf{\Pi}) \cdot (\mathbf{\Gamma} \times \delta\mathbf{\Theta}, \delta\mathbf{\Pi})}$$

for all $(\mathbf{\Gamma} \times \delta\mathbf{\Theta}, \delta\mathbf{\Pi}) \in T_{(\mathbf{\Gamma}, \mathbf{\Pi})}\mathcal{P}_\mu$.

Example: the Heavy Top

The reduced symplectic form on $\mathcal{P}_\mu$ is

$$\boxed{\omega_\mu^{\mathcal{P}}(\mathbf{\Gamma}, \mathbf{\Pi})\left((\mathbf{\Gamma} \times \mathbf{X}_\Theta, \mathbf{X}_\Pi), (\mathbf{\Gamma} \times \delta\mathbf{\Theta}, \delta\mathbf{\Pi})\right) = (\mathbf{\Pi} \times \mathbf{X}_\Theta - \mathbf{X}_\Pi) \cdot \delta\mathbf{\Theta} + \mathbf{X}_\Theta \cdot}$$

while the directional derivative of the reduced Hamiltonian H_μ in the direction $(\mathbf{\Gamma} \times \delta\mathbf{\Theta}, \delta\mathbf{\Pi}) \in T_{(\mathbf{\Gamma},\mathbf{\Pi})}\mathcal{P}_\mu \simeq \mathbb{R}^3 \times \mathbb{R}^3$ is

$$\boxed{DH_\mu(\mathbf{\Gamma}, \mathbf{\Pi}) \cdot (\mathbf{\Gamma} \times \delta\mathbf{\Theta}, \delta\mathbf{\Pi}) = -mg\ell(\mathbf{\Gamma} \times \mathbf{e_3}) \cdot \delta\mathbf{\Theta} + \mathbf{\Omega} \cdot \delta\mathbf{\Pi}.}$$

We find eventually the reduced Hamiltonian vector field

$$X_{H_\mu}(\mathbf{\Gamma}, \mathbf{\Pi}) = (\mathbf{\Gamma} \times \mathbf{X}_\Theta, \mathbf{X}_\Pi) = (\mathbf{\Gamma} \times \mathbf{\Omega}, \mathbf{\Pi} \times \mathbf{\Omega} + mg\ell(\mathbf{\Gamma} \times \mathbf{e_3}))\,.$$

Example: the Heavy Top

The reduced equations of motion on P_μ are then given by

$$\begin{cases} \dot{\mathbf{\Gamma}} & = \mathbf{\Gamma} \times \mathbf{\Omega} \\ \dot{\mathbf{\Pi}} & = \mathbf{\Pi} \times \mathbf{\Omega} + mg\ell(\mathbf{\Gamma} \times \mathbf{e_3}) \end{cases}$$

where $\| \mathbf{\Gamma} \| = 1$ and $\mathbf{\Pi} \cdot \mathbf{\Gamma} = \mu$ for $\mathbf{\Gamma}, \mathbf{\Pi} \in \mathbb{R}^3$.

References.

- *Hamiltonian Reduction by Stages* by J. E. Marsden, G. Misiolek, J.-P. Ortega, M. Perlmutter and T. S. Ratiu 2009
- *Foundations of mechanics* by R. Abraham and J. E. Marsden 1987
- *Global aspects of classical integrable systems* by R. H. Cushman and L.M. Bates 2004
- *The heavy top: a geometric treatment* by D. Lewis, T. Ratiu, J.C. Simo and J.E. Marsden 1991
- *Mathematical methods of classical mechanics* by V.I. Arnold 1988
- *Reduction of symplectic manifolds with symmetry* by J. Marsden and A. Weinstein 1974