Complexity of Real Root Isolation Using Continued Fractions

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![Diagram showing the graph of $A(X)$ with roots at $c$ and $d$.]
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Fundamental Task

Computer Algebra, Computational Geometry, Quantifier Elimination etc.
Real Root Isolation

Let $A(X)$ be a polynomial with coefficients in $\mathbb{R}$.

A($X$) is a square-free polynomial of degree $n$.

Fundamental Task

Computer Algebra, Computational Geometry, Quantifier Elimination etc.
A General Subdivision Algorithm for Real Root Isolation
Estimate on number of real roots

- $E(A, (c,d))$ an upper bound on number of real roots of $A(X)$ in $(c,d)$.
- If $E(A, (c,d)) = 1$ then there is exactly one real root of $A(X)$ in $(c,d)$. 
### A General Subdivision Algorithm for Real Root Isolation

#### Estimate on number of real roots

- \( E(A, (c, d)) \) an upper bound on number of real roots of \( A(X) \) in \((c, d)\).
- If \( E(A, (c, d)) = 1 \) then there is exactly one real root of \( A(X) \) in \((c, d)\).

#### Input:
Polynomial \( A(X) \in \mathbb{R}[X] \) and \((c, d) \subseteq \mathbb{R}\).

#### Output:
List of isolating intervals for real roots of \( A(X) \) in \((c, d)\).
A General Subdivision Algorithm for Real Root Isolation

Estimate on number of real roots

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Output: List of isolating intervals for real roots of \( A(X) \) in \( (c, d) \).

\[ \text{RootIsol}(A, (c, d)) \]

1. If \( E(A, (c, d)) = 0 \) return.
2. If \( E(A, (c, d)) = 1 \) output \((c, d)\).
3. Partition \((c, d)\) into two intervals \( I, J \).
4. \( \text{RootIsol}(A, I) \) and \( \text{RootIsol}(A, J) \).
A General Subdivision Algorithm for Real Root Isolation

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How to implement \( E(A, (c, d)) \)?
Two Varieties of Real Root Isolation Algorithm

Sturm Sequences

1. \( E(A, (c, d)) \) is computed from the Sturm sequence of \( A(X), A'(X) \).
2. \( E(A, (c, d)) = \) number of real roots of \( A(X) \) in \((c, d)\).
### Sturm Sequences

1. $E(A, (c, d))$ is computed from the Sturm sequence of $A(X), A'(X)$.
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### The Descartes’ Rule of Signs

1. $E(A, (c, d)) = \text{sign variation in the Bernstein coeffs. of } A(X) \text{ w.r.t. } (c, d)$.
2. $E(A, (c, d)) \geq \text{number of real roots of } A(X) \text{ in } (c, d) \text{ by a +ve even number.}$
Two Varieties of Real Root Isolation Algorithm

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In practice, the second approach is more efficient than the first one.
Vincent’s Theorem, 1836

Transform $A(X)$ as follows:
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Transform \( A(X) \) as follows:

\[
A(X) \rightarrow X^n A \left( a_0 + \frac{1}{X} \right)
\]

\( a_0 \in \mathbb{N}_{\geq 0}, \)
Transform $A(X)$ as follows:

\[ A(X) \rightarrow X^n A \left( a_0 + \frac{1}{X} \right) \rightarrow (a_1 X + 1)^n A \left( a_0 + \frac{1}{a_1 + \frac{1}{X}} \right) \]

\[ a_0 \in \mathbb{N}_{\geq 0}, \ a_1 \in \mathbb{N}_{> 0}, \]
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\rightarrow (a_1 a_2 X + a_1 + 1)^n A \left( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{X}}} \right)
\]

$a_0 \in \mathbb{N}_{\geq 0}$, $a_1 \in \mathbb{N}_{>0}$, $a_2 \in \mathbb{N}_{>0}$. 
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Transform $A(X)$ as follows:

$$A(X) \rightarrow X^n A\left(a_0 + \frac{1}{X}\right) \rightarrow (a_1 X + 1)^n A\left(\frac{1}{a_1} + \frac{1}{X} \right)$$

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$a_0 \in \mathbb{N}_{\geq 0}$, $a_1 \in \mathbb{N}_{> 0}$, $a_2 \in \mathbb{N}_{> 0}$.

Resulting polynomial has at most one sign variation in its coefficients.
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Resulting polynomial has at most one sign variation in its coefficients.

Let $\text{Var}(A)$ be the number of sign variations in the coefficients of $A(X)$.
We want to isolate the positive roots of $A(X)$. 
Construct $A_R(X) := A(X + 1)$, $M_R(X) := X + 1$. Check if $\text{Var}(A_R)$ is 0 or 1.
Construct $A_L(X) := (X + 1)^n A \left(\frac{1}{X+1}\right)$, $M_L(X) := (X + 1)^{-1}$. 
Check if $\text{Var}(A_L)$ is 0 or 1.
Construct $A_{RR}(X) := A_R(X + 1) = A(X + 2)$, $M_{RR}(X) := X + 2$. 
Check if $\text{Var}(A_{RR})$ is 0 or 1.
Construct $A_{RL}(X) := A_R\left(\frac{1}{1+X}\right) = A\left(1 + \frac{1}{1+X}\right)$ and $M_{RL}(X) := 1 + \frac{1}{X+1}$. 
Check $\text{Var}(A_{RL})$. 
\[ A_{LR}(X) = A_L(X + 1) = (X + 2)^n A \left( 1 + \frac{1}{2+X} \right), \quad M_{LR}(X) := (X + 2)^{-1} \]
\[
\text{Var}(A_{LR}) = 1, \text{ return } M_{LR}(0) = \frac{1}{2}, M_{LR}(\infty) = 0
\]
\[ A_{RL}(X) := (X + 1)^n A_L \left( \frac{1}{1+X} \right) = (X + 2)^n A \left( \frac{1}{1+\frac{1}{1+X}} \right), \quad M_{LR}(X) := \frac{X+1}{X+2} \]
\[ \text{Var}(A_{RL}) = 1, \text{ return } M_{RL}(0) = \frac{1}{2}, M_{RL}(\infty) = 1 \]
Continue recursively at each level
This was Uspensky’s algorithm [Uspensky, 1948].
Vincent’s Algorithm for Isolating Positive Roots

We want to isolate the positive roots of $A(X)$.
Construct $A_R(X) := A(X + 1)$, $M_R(X) := X + 1$ and check if $\text{Var}(A_R)$ is 0 or 1.
Vincent’s Algorithm for Isolating Positive Roots

Is $\text{Var}(A_R) < \text{Var}(A)$?
Construct $A_L(X) := (X + 1)^n A \left( \frac{1}{X+1} \right)$, $M_L(X) := (X + 1)^{-1}$.
Vincent’s Algorithm for Isolating Positive Roots

Continue recursively at each level
If $\text{Var}(A_R) = \text{Var}(A)$ then don’t construct $A_L(X)$. 

Vincent’s Algorithm for Isolating Positive Roots
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But proceed recursively from $A_R(X)$.
Vincent’s Algorithm for Isolating Positive Roots

If \( \text{Var}(A_R) = \text{Var}(A) \) then don’t construct \( A_L(X) \).

Budan-Fourier

\[
\#(\text{roots in } (0, 1)) \leq \text{Var}(A(X)) - \text{Var}(A(X + 1)) = \text{Var}(A) - \text{Var}(A_R).
\]
Drawback of Vincent’s Algorithm

Exponential running time

- Consider the polynomial \( A(X) = (X - 2^L)(X - 2^L - 1) \).
Drawback of Vincent’s Algorithm

Exponential running time

- Consider the polynomial \( A(X) = (X - 2^L)(X - 2^L - 1) \).
- At depth \( i \) on the right most path the polynomial is
  \[
  A(X + i) = (X - (2^L - i))(X - (2^L + 1 - i)).
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Drawback of Vincent’s Algorithm

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- To get the smallest positive root of \( A(X) \) in the unit interval \( i \geq 2^L \).
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Two Solutions

- [Collins/Akritas, 1976]: Bisect the interval at each recursion level.
### Drawback of Vincent’s Algorithm

**Exponential running time**

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- [Collins/Akritas, 1976]: Bisect the interval at each recursion level.
- [Akritas, 1978]: Can we do better than shifts by unit length? 
  
  Idea: Use a lower bound on the smallest positive root.
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Exponential running time

- Consider the polynomial $\displaystyle A(X) = (X - 2^L)(X - 2^L - 1)$.
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- [Akritas, 1978]: Can we do better than shifts by unit length?
  Idea: Use a lower bound on the smallest positive root.

Advantages of Akritas’ approach

- Faster in practice.
- Utilises distribution of roots.
- Computes the continued fraction approximation of the roots.
Akritas’ Algorithm

Input: Polynomial $A(X)$ of degree $n$ whose coefficients are real numbers.
Output: List of isolating intervals for the positive roots of $A(X)$. 
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$\text{RootIsol}(A, M)$

- If $\text{Var}(A) = 0$ return.
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$\text{RootIsol}(A, M)$

- If $\text{Var}(A) = 0$ return.
- If $\text{Var}(A) = 1$ output the interval with end points $M(0), M(\infty)$. 
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- If $\text{Var}(A) = 0$ return.
- If $\text{Var}(A) = 1$ output the interval with end points $M(0), M(\infty)$.
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- If $B \geq 1$ then $A(X) := A(X + B), M(X) := M(X + B)$. 
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- Compute $A_R(X) := A(X + 1)$ and $M_R(X) := M(X + 1)$.
- If $\text{Var}(A_R) < \text{Var}(A)$ then $A_L(X) := (X + 1)^n A\left(\frac{1}{X+1}\right)$, $M_L(X) := M\left(\frac{1}{X+1}\right)$. 
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- If $\text{Var}(A_R) < \text{Var}(A)$ then $A_L(X) := (X + 1)^n A\left(\frac{1}{X+1}\right), M_L(X) := M\left(\frac{1}{X+1}\right)$.
- $RootIsol(A_R, M_R)$ and $RootIsol(A_L, M_L)$. 
Worst case bit-complexity of Akritas’ algorithm?
### Two steps for getting the worst-case bounds

1. **Bound the worst-case size of the recursion tree:**
   - number of inversion transformations, $X \rightarrow (X + 1)^{-1}$ and
   - number of Taylor shifts.

2. **Bound the worst-case complexity of a node in the recursion tree.**
Worst case bit-complexity of Akritas’ algorithm?

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Akritas’ worst case bit-complexity

For $A(X) \in \mathbb{Z}[X]$, degree $n$, coefficients of bit-length $L$ – $\tilde{O}(n^4L^2)$:

- number of inversion transformations and Taylor shifts – $\tilde{O}(n^2L)$.
- worst case bit-complexity of a node using fast integer arithmetic – $\tilde{O}(n^2L)$. 
Worst case bit-complexity of Akritas’ algorithm?

Two steps for getting the worst-case bounds

1. Bound the worst-case size of the recursion tree:
   - number of inversion transformations, \( X \to (X + 1)^{-1} \) and
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Akritas’ worst case bit-complexity

For \( A(X) \in \mathbb{Z}[X] \), degree \( n \), coefficients of bit-length \( L \sim O(n^4L^2) \):

- number of inversion transformations and Taylor shifts \( \sim O(n^2L) \).
- worst case bit-complexity of a node using fast integer arithmetic \( \sim O(n^2L) \).

Drawbacks

- Assumes floor of the smallest positive root can be computed in \( O(1) \).
- Assumes Taylor shifts don’t increase the bit-size.
Worst case bit-complexity of Akritas’ algorithm?

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Akritas’ worst case bit-complexity

For \( A(X) \in \mathbb{Z}[X] \), degree \( n \), coefficients of bit-length \( L \) – \( \tilde{O}(n^4L^2) \):

- number of inversion transformations and Taylor shifts – \( \tilde{O}(n^2L) \).
- worst case bit-complexity of a node using fast integer arithmetic – \( \tilde{O}(n^2L) \).

Our worst case bit-complexity

Worst case bit-complexity is \( \tilde{O}(n^7L^2) \):

- number of inversion transformations \( \tilde{O}(nL) \); no. of Taylor shifts \( \tilde{O}(n^3L) \).
- worst case bit-complexity of a node using fast integer arithmetic.
Akritas’ Algorithm

**RootIsol**(*A*, *M*)

- If $\text{Var}(A) = 0$ return.
- If $\text{Var}(A) = 1$ output the interval with end points $M(0)$, $M(\infty)$.
- Compute a lower bound $B$ on the positive roots of $A(X)$.
- If $B \geq 1$ then $A(X) := A(X + B)$, $M(X) := M(X + B)$.
- Compute $A_R(X) := A(X + 1)$ and $M_R(X) := M(X + 1)$.
- If $\text{Var}(A_R) < \text{Var}(A)$ then $A_L(X) := (X + 1)^n A\left(\frac{1}{X+1}\right)$, $M_L(X) := M\left(\frac{1}{X+1}\right)$.

• What are the transformations $M_R$, $M_L$?
• What is the relation between $A_R$, $A_L$ and the input polynomial?
The transformation associated with a node in the tree
The transformation associated with a node in the tree

- Transformation associated with root is $X$. 
The transformation associated with a node in the tree

- Transformation associated with root is $X$.
- Do a Taylor shift $X \rightarrow X + a$. 
The transformation associated with a node in the tree

- Transformation associated with root is $X$.
- Do a Taylor shift $X \rightarrow X + a$.
- Then the transformation $X \rightarrow \frac{1}{1+X}$.
The transformation associated with a node in the tree

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- Then the transformation $X \rightarrow \frac{1}{1+X}$.
- Then a Taylor shift $X \rightarrow X + b_0$. 
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- Then the transformation $X \rightarrow \frac{1}{1+X}$.
- Then a Taylor shift $X \rightarrow X + b_0$.
- Again a Taylor shift by $X \rightarrow X + b_1$. 
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- Followed by $X \rightarrow \frac{1}{1+X}$.
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- Again a Taylor shift by $X \rightarrow X + b_1$.
- Followed by $X \rightarrow \frac{1}{1+X}$.
- Associated transformation is
  \[ a + \frac{1}{1+b_0+b_1+\frac{1}{1+X}} \]
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- Associated transformation is
  \[ a + \frac{1}{1+b_0+b_1+\frac{1}{1+X}} \]
- Same as $a + \frac{1}{q+\frac{1}{1+X}}$, $q = 1 + b_0 + b_1$. 

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Real Root Isolation – Continued Fractions
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- Then the transformation $X \rightarrow \frac{1}{1+X}$.
- Then a Taylor shift $X \rightarrow X + b_0$.
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- Followed by $X \rightarrow \frac{1}{1+X}$.
- Associated transformation is $a + \frac{1}{1+b_0+b_1+\frac{1}{1+X}}$.
- Same as $a + \frac{1}{q+\frac{1}{1+X}}$, $q = 1 + b_0 + b_1$.
- Collapse consecutive Taylor shifts into one.
The transformation associated with a node in the tree

- Transformation associated with root is $X$.
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- Associated transformation is $a + \frac{1}{1+b_0+b_1+\frac{1}{1+X}}$
- Same as $a + \frac{1}{q+\frac{1}{1+X}}$, $q = 1 + b_0 + b_1$.
- Collapse consecutive Taylor shifts into one.

What is the transformation in general?
The transformation associated with a node in the tree

\[
q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{\ddots + \frac{1}{q_m + X}}}}
\]

where

- \( m \) is the number of inversion transformations \((X \rightarrow \frac{1}{1+X})\).
- \( q_0 \geq 0 \) the total amount of Taylor shifts to the first inversion transformation.
- \( q_i \geq 1 \), for \( i = 1, \ldots, m - 1 \), the total amount of Taylor shifts between \( i \)-th and \( i + 1 \)-th inversion transformation; if there are no Taylor shifts \( q_i = 1 \).
The transformation associated with a node in the tree

\[ q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{\ddots + \frac{1}{q_m + X}}}} \]

where

- \( m \) is the number of inversion transformations \((X \rightarrow \frac{1}{1+X})\).
- \( q_0 \geq 0 \) the total amount of Taylor shifts to the first inversion transformation.
- \( q_i \geq 1, \text{ for } i = 1, \ldots, m - 1 \), the total amount of Taylor shifts between \( i \)-th and \( i + 1 \)-th inversion transformation; if there are no Taylor shifts \( q_i = 1 \).

Let \( i \)-th quotient \( \frac{P_i}{Q_i} \) be the finite continued fraction \( q_0 + \frac{1}{q_1 + \frac{1}{\ddots + \frac{1}{q_i}}} \).

Then \( P_i = q_iP_{i-1} + P_{i-2} \) and \( Q_i = q_iQ_{i-1} + Q_{i-2} \).
The transformation associated with a node in the tree

\[
q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \cdots + \frac{1}{q_m + X}}} = \frac{P_mX + P_{m-1}}{Q_mX + Q_{m-1}}.
\]

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Two additional features associated with a node in the tree

The transformation associated with a node

Let $m$ be the number of inversion transformations along the path and

$$M(X) := \frac{P_m X + P_{m-1}}{Q_m X + Q_{m-1}}.$$
Two additional features associated with a node in the tree

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Let \( m \) be the number of inversion transformations along the path and

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The two features

- Polynomial \( A_m(X) := (Q_m X + Q_{m-1})^n A(M(X)) \).
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- Polynomial $A_m(X) := (Q_mX + Q_{m-1})^n A(M(X))$.
- Interval $I_m$ that has end-points $M(0) = \frac{P_{m-1}}{Q_{m-1}}, M(\infty) = \frac{P_m}{Q_m}$.

Note: Width of $I_m$ is $\left| \frac{P_m}{Q_m} - \frac{P_{m-1}}{Q_{m-1}} \right| = (Q_mQ_{m-1})^{-1}$. 
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The positive roots of $A_m(X) \Leftrightarrow$ Roots of $A(X)$ in $I_m$.

$\text{Var}(A_m) = \#(\text{number of roots of } A(X) \text{ in } I_m) + \text{even number}$. 
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When does the algorithm terminate? When is \( \text{Var}(A_m) \leq 1? \)
Termination Criterion: Two-Circle Theorem

$\mathcal{I}_m$
Termination Criterion: Two-Circle Theorem

\[ I_m \]
Termination Criterion: Two-Circle Theorem

\[ \text{I}_m \]
Termination Criterion: Two-Circle Theorem

Two-circle Theorem ([Ostrowski, 1950])

If the two-circles figure w.r.t. $I_m$ contains a single root of $A(X)$ then $\text{Var}(A_m) = 1$; if no roots of $A(X)$ then $\text{Var}(A_m) = 0$. 
Two-circle Theorem ([Ostrowski, 1950])

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Contrapositive

If $\text{Var}(A_m) \geq 2$, then the two-circles figure in $\mathbb{C}$ w.r.t. interval $I_m$ contains two roots $\alpha, \beta$ of $A(X)$.
Termination Criterion: Two-Circle Theorem

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Corollary

*We can choose a pair $\alpha, \beta$ of roots inside the two-circles such that*
Termination Criterion: Two-Circle Theorem

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Corollary

We can choose a pair $\alpha, \beta$ of roots inside the two-circles such that 
$$|\beta - \alpha| < \sqrt{3}|I_m|$$
Termination Criterion: Two-Circle Theorem

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If $\text{Var}(A_m) \geq 2$, then the two-circles figure in $\mathbb{C}$ w.r.t. interval $I_m$ contains two roots $\alpha, \beta$ of $A(X)$.

Corollary

We can choose a pair $\alpha, \beta$ of roots inside the two-circles such that $|\beta - \alpha| < \sqrt{3}|I_m|$, but $|I_m| = \left| \frac{P_m}{Q_m} - \frac{P_{m-1}}{Q_{m-1}} \right| = \frac{1}{Q_mQ_{m-1}}$. Thus
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We can choose a pair $\alpha, \beta$ of roots inside the two-circles such that $|\beta - \alpha| < \sqrt{3}|I_m|$, but $|I_m| = \left| \frac{P_m}{Q_m} - \frac{P_{m-1}}{Q_{m-1}} \right| = \frac{1}{Q_mQ_{m-1}}$. Thus

If $\text{Var}(A_m) \geq 2$ then $\frac{1}{Q_mQ_{m-1}} > |\beta - \alpha|/\sqrt{3}$. 
A path in the recursion tree of \( \text{RootIsol}(A,X) \), from the root to a parent \( J \) of two leaves. Let \( m \) be the number of inversion transformations along the path.
Number of Inversion Transformations along a path to a leaf

1. A path in the recursion tree of $\text{RootIsol}(A, X)$, from the root to a parent $J$ of two leaves. Let $m$ be the number of inversion transformations along the path.

2. Let $I_m$ be the interval associated with $J$. 

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Real Root Isolation – Continued Fractions
A path in the recursion tree of $\text{RootIsol}(A, X)$, from the root to a parent $J$ of two leaves. Let $m$ be the number of inversion transformations along the path.

Let $I_m$ be the interval associated with $J$.

Since $\text{Var}(A_m) \geq 2$, there is a pair of roots $(\alpha_J, \beta_J)$ of $A(X)$ such that

$$|I_m| = \frac{1}{Q_m Q_{m-1}} \geq |\beta_J - \alpha_J| / \sqrt{3}.$$
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1. A path in the recursion tree of $\text{RootIsol}(A, X)$, from the root to a parent $J$ of two leaves. Let $m$ be the number of inversion transformations along the path.

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$$Q_m = q_m Q_{m-1} + Q_{m-2} \geq Q_{m-1} + Q_{m-2} \geq F_m \geq \phi^{m-1}.$$
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5. Thus \( m \leq 2 - \log_\phi |\beta_J - \alpha_J| \).

6. This was shown by Uspensky and Ostrowski.
Number of Inversion Transformations along a path to a leaf

1. A path in the recursion tree of RootIsol\( (A, X) \) from the root to a parent \( J \) of two leaves. Let \( m \) be the number of inversion transformations along the path.

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Proposition

The total number of inversion transformations in the tree are bounded by

\[
\sum_J (2 - \log_\phi |\beta_J - \alpha_J|).
\]
Akritas’ Algorithm

\textbf{RootIsol}(A,M)

- If $\text{Var}(A) = 0$ return.
- If $\text{Var}(A) = 1$ output the interval with end points $M(0)$, $M(\infty)$.
- Compute a lower bound $B$ on the positive roots of $A(X)$.
- If $B \geq 1$ then $A(X) := A(X + B)$, $M(X) := M(X + B)$.
- Compute $A_R(X) := A(X + 1)$ and $M_R(X) := M(X + 1)$.
- If $\text{Var}(A_R) < \text{Var}(A)$ then $A_L(X) := (X + 1)^n A\left(\frac{1}{X+1}\right)$, $M_L(X) := M\left(\frac{1}{X+1}\right)$.
- $\text{RootIsol}(A_R, M_R)$ and $\text{RootIsol}(A_L, M_L)$. 
Akritas’ Algorithm

RootIsol\((A, M)\)

- If \(\text{Var}(A) = 0\) return.
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- \(\text{RootIsol}(A_R, M_R)\) and \(\text{RootIsol}(A_L, M_L)\).

How do we compute a lower bound on positive roots of a polynomial?
Lower Bound on the smallest positive root

One Approach

- Roots of $X^nA(1/X)$ are inverse of the roots of $A(X)$.
- Compute an upper bound $U$ on the largest positive root of $X^nA(1/X)$.
- $1/U$ is a lower bound on the smallest positive root of $A(X)$. 
Lower Bound on the smallest positive root

One Approach

- Roots of $X^n A(1/X)$ are inverse of the roots of $A(X)$.
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- $1/U$ is a lower bound on the smallest positive root of $A(X)$.

Upper bound on positive roots, [Hong,98]

$$B(X) = \sum_{i=0}^{n} b_i X^i, \ b_n > 0. \ U(B) := 2 \max_{b_i < 0} \min_{b_j > 0, j > i} \left\{ \left| \frac{b_i}{b_j} \right|^{1/(j-i)} \right\}.$$
Lower Bound on the smallest positive root

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Tight lower bound

Define $\text{PLB}(A) := \frac{1}{U(X^n A(1/X))}$. Suppose $A(X)$ has only real roots in $\Re(z) > 0$ and $\alpha$ is the smallest positive root of $A(X)$. Then

$$\frac{\alpha}{2^n} \leq \text{PLB}(A) < \alpha.$$
Bound on the number of Taylor shifts along a path

Assume the polynomial $A(X)$ has *only real roots*.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?
Bound on the number of Taylor shifts along a path

Assume the polynomial $A(X)$ has only real roots.

$(\text{shifts needed to reach the floor of the smallest positive root } \alpha_0)$?
Assume the polynomial $A(X)$ has *only real roots*.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

Since $\text{PLB}(A)$ is a lower bound on $\alpha_0$ we have $\frac{|\alpha_0|}{2n} \leq \text{PLB}(A)$. 
Assume the polynomial $A(X)$ has only real roots.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

Shift $A(X)$ by $\text{PLB}(A)$ to obtain $A_1(X)$. 

$A(X) = A(X + a_0)$
Bound on the number of Taylor shifts along a path

Assume the polynomial $A(X)$ has *only real roots*.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

\[ a = \text{PLB}(A) \]

Suppose $\alpha_1 > 1$. Then \( \frac{\alpha_1}{2n} \leq \text{PLB}(A_1) \).
Assume the polynomial $A(X)$ has *only real roots*.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

Shift $A_1(X)$ by $a_1$ to get $A_2(X)$. 

$A_2(X) = A_1(X + a_1)$
Assume the polynomial $A(X)$ has *only real roots*.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

\[ \alpha_2 = \alpha_1 - a_1 \]

Again suppose $\alpha_2 > 1$. Then $\frac{\alpha_2}{2n} \leq \text{PLB}(A_2)$. 

\[ \beta_2 = \beta_1 - a_1 \]

\[ \gamma_2 = \gamma_1 - a_1 \]

\[ a = \text{PLB}(A) \]

\[ A_2(X) = A_1(X+a_1) \]
Assume the polynomial $A(X)$ has only real roots.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

$$A_3(X) = A_2(X + a_2)$$

This continues until $\alpha_i < 1$, i.e., we compute the floor of $\alpha_0$. 
Assume the polynomial $A(X)$ has *only real roots*. 

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

This continues until $\alpha_i < 1$, i.e., we compute the floor of $\alpha_0$.

- $\alpha_i = \alpha_{i-1} - \text{PLB}(A_{i-1}) \leq \alpha_{i-1}(1 - \frac{1}{2^n})$. 

\[
\begin{align*}
\beta_3 &= \beta_2 - a_2 \\
\alpha_3 &= \alpha_2 - a_2 \\
\gamma_3 &= \gamma_2 - a_2
\end{align*}
\]
Assume the polynomial $A(X)$ has \textit{only real roots}. 

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

This continues until $\alpha_i < 1$, i.e., we compute the floor of $\alpha_0$.

- $\alpha_i = \alpha_{i-1} - \text{PLB}(A_{i-1}) \leq \alpha_{i-1}(1 - \frac{1}{2n})$.
- Thus $\alpha_i \leq \alpha_0 (1 - \frac{1}{2n})^i$. 
Bound on the number of Taylor shifts along a path

Assume the polynomial $A(X)$ has only real roots.

#(shifts needed to reach the floor of the smallest positive root $\alpha_0$)?

This continues until $\alpha_i < 1$, i.e., we compute the floor of $\alpha_0$.

- $\alpha_i = \alpha_{i-1} - \text{PLB}(A_{i-1}) \leq \alpha_{i-1}(1 - \frac{1}{2n})$.
- Thus $\alpha_i \leq \alpha_0(1 - \frac{1}{2n})^i$.
- Need at most $2n \log \alpha_0$ Taylor shifts to compute floor of $\alpha_0$. 

\[\begin{align*}
\beta_3 &= \beta_2 - a_2 \\
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Number of Taylor shifts along a path in the tree

Assume the polynomial $A(X)$ has *only real roots*.

1. Consider a path in the recursion tree of $\text{RootIsol}(A, M(X))$, $M(X) = X$, from the root to a parent $J$ of two leaves.

2. Let $\alpha_J, \beta_J$ be the roots associated with the leaves.

3. $m$ be the number of inversion transformations along the path.
Number of Taylor shifts along a path in the tree

Assume the polynomial $A(X)$ has *only real roots*.

Consider the $i$-th and $i + 1$-th inversion transformation.

$A_i(X)$ be the polynomial associated with the blue node. $a_1, \ldots, a_\ell$ be its positive real roots.

#(Taylor shift) from $i$-th to $i + 1$-th transformation is bounded by

$$2n (\log a_1 + \cdots + \log a_\ell) \leq 2n^2 \log a_\ell \leq 2n^2 \log q_i.$$
Number of Taylor shifts along a path in the tree

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Assume the polynomial $A(X)$ has only real roots.

Consider the $i$-th and $i+1$-th inversion transformation.

Let $A_i(X)$ be the polynomial associated with the blue node. Let $a_1, \ldots, a_{\ell}$ be its positive real roots.

The number of Taylor shifts $(\text{Taylor shift})$ from $i$-th to $i+1$-th transformation is bounded by

$$2n(\log a_1 + \cdots + \log a_{\ell}) \leq 2n^2 \log a_{\ell} \leq 2n^2 \log q_i.$$
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Total number of Taylor shifts on the path to $J$ is $n^2 O(\sum_{i=1}^{m} \log q_i)$. 

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Number of Taylor shifts along a path in the tree

Assume the polynomial $A(X)$ has *only real roots*.

Total number of Taylor shifts on the path to $J$ is $n^2 \mathcal{O}(\sum_{i=1}^{m} \log q_i)$.

We can show

\[ \sum_{i=1}^{m} \log q_i = \mathcal{O}(\log |\alpha_j - \beta_j|^{-1}) \]
Number of Taylor shifts along a path in the tree

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We can show

\[\sum_{i=1}^{m} \log q_i = O\left(\log |\alpha_J - \beta_J|^{-1}\right)\]

Total number of Taylor shifts on the path to $J$ is

\[n^2 O\left(\log |\alpha_J - \beta_J|^{-1}\right)\]
Number of Taylor shifts along a path in the tree

Assume the polynomial \( A(X) \) has only real roots.

**Proposition**

The total number of Taylor shifts in the tree is bounded by

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Lower bound on \( \prod_J |\alpha_J - \beta_J| \)?

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**Theorem (Davenport–Mahler [Dav., 1985] [Du/Sharma/Yap, 2005])**

Consider a polynomial $A(X) \in \mathbb{C}[X]$ of degree $n$. Let $G = (V, E)$ be a DAG whose vertices are the roots of $A(X)$. If

(i) $(\alpha, \beta) \in E \implies |\alpha| \leq |\beta|$, and

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then

$$\prod_{(\alpha, \beta) \in E} |\beta - \alpha| \geq \frac{\sqrt{|\text{discr}(A)|}}{M(A)^{n-1}} \cdot 2^{-O(n \log n)},$$

where, if $\vartheta_i$ are roots of $A(X)$,

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**Corollary**

If $A(X) \in \mathbb{Z}[X]$ is square-free, has degree $n$, and coefficient bit-length $L$ then

$$\prod_{(\alpha, \beta) \in E} |\beta - \alpha| = 2^{-O(nL)}.$$
Assume $A(X)$ has degree $n$, coefficient bit-length $L$ and only real roots.
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- The result holds in general!
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Vikram Sharma (INRIA, Sophia-Antipolis)
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Combined with our worst-case bound $\tilde{O}(n^3L)$ on tree-size.
Main Result

Theorem

For a square-free integer polynomial of degree \( n \), and coefficients of bit-length \( L \), the worst-case running time of Akritas’ algorithm is bounded by \( \tilde{O}(n^7 L^2) \).
Expected Complexity, [Tsigaridas/Emiris, 2006]

The size of the tree

- Assumes floor of the smallest positive root can be computed in $O(1)$.
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Expected complexity of a node

- From Khinchin’s result we know \( E[\sum_{j=0}^{i} \log q_j] = i + 1 = \widetilde{O}(nL) \).
- Expected cost at a node is \( O(n^2M(L + n\sum_{j=0}^{i} b_i)) = \widetilde{O}(n^4L) \).
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Theorem

Expected running time of Akritas’ algorithm:

- $\widetilde{O}(n^5L^2)$ using classical Taylor shifts with fast integer arithmetic,
- $\widetilde{O}(n^4L^2)$ using asymptotically fast Taylor shifts.
Comparison with the Descartes method

Input: $A(X) \in \mathbb{R}[X]$ of degree $n$, and $(c, d)$.
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Vikram Sharma (INRIA, Sophia-Antipolis)
Real Root Isolation – Continued Fractions
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Output: Isolating intervals for roots of $A(X)$ in $(c, d)$.

Estimate on the number of roots, $E(A, (c, d))$

Let $A(X) = \sum_{i=0}^{n} a_i \binom{n}{i} (X - c)^i (d - X)^{n-i} (d - c)^{-n}$.
$E(A, (c, d)) := \#(\text{sign variations in } (a_n, a_{n-1}, \ldots, a_0)).$

- If $E(A, (c, d)) = 0$ then $A(X)$ has no roots in $(c, d)$.
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But...

- Degree 100 Mignotte’s polynomials ($X^n - (aX - 1)^2$):
  [Emiris/Tsigaridas, ’06]: Descartes 7.83sec. and Akritas 0.02sec.
- Available in Mathematica.
- [Collins/Akritas, 1976]: $O(n^6L^2)$; [Johnson, 1998]: $O(n^4L^2)$. 
Possible ways to improve the complexity

1. Derive tight bounds on largest positive root of a polynomial in $O(n)$ operations. Bounds by [Kioustelidis, 1986; Ştefănescu, 2005] are known to be not tight. A recent bound by Akritas et al. might help.
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Open question

For Mignotte’s polynomial $X^n - 2(aX - 1)^2$, $a \in \mathbb{N}$, the size of the recursion tree is $O(\log a)$ using Zassenhaus’ bound (the Descartes method has recursion tree size $\Omega(n \log a)$).
Summary

Main Result

For a square-free polynomial $A(X)$, degree $n$, and coefficient bit-length $L$:

1. Worst case bit-complexity of Akritas’ algorithm is $\tilde{O}(n^7L^2)$. 
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Merci!