Complexity of Real Root Isolation Using Continued Fractions

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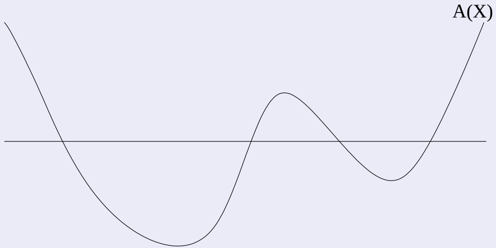
Real Root Isolation

Let $A(X)$ be a polynomial with coefficients in $\mathbb{R}$.



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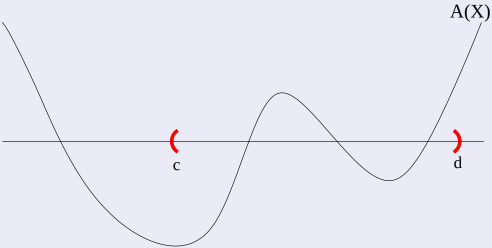

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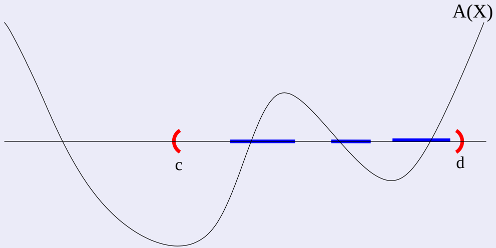

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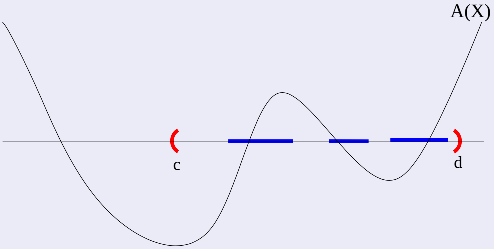

Fundamental Task

Computer Algebra, Computational Geometry, Quantifier Elimination etc.

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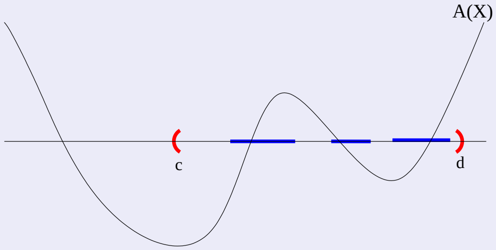

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$A(X)$ is a *square-free* polynomial of degree n.

A General Subdivision Algorithm for Real Root Isolation

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Estimate on number of real roots

- $E(A,(c,d))$ an *upper bound* on number of real roots of $A(X)$ in (c,d).
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1. **If** $E(A,(c,d)) = 0$ **return**.
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How to implement $E(A,(c,d))$?

Two Varieties of Real Root Isolation Algorithm

Sturm Sequences

1. $E(A,(c,d))$ is computed from the Sturm sequence of $A(X), A'(X)$.
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1. $E(A,(c,d))=$ sign variation in the Bernstein coeffs. of $A(X)$ w.r.t. (c,d).
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In practice, the second approach is more efficient than the first one.

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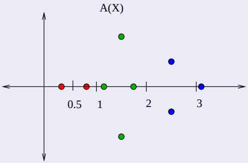
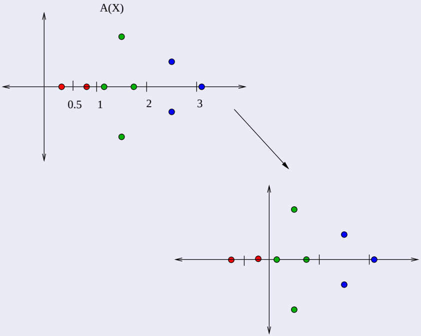
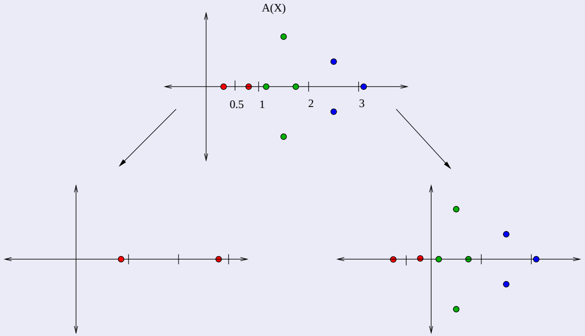
Let $\mathrm{Var}(A)$ be the number of sign variations in the coefficients of $A(X)$.



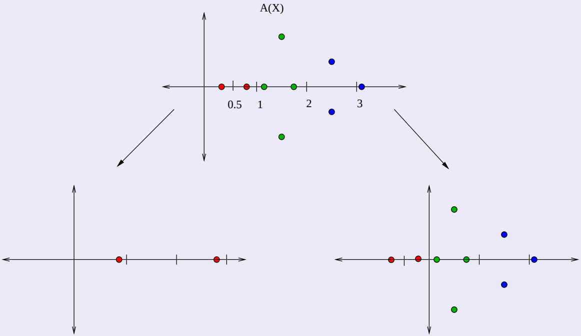
We want to isolate the positive roots of $A(X)$.



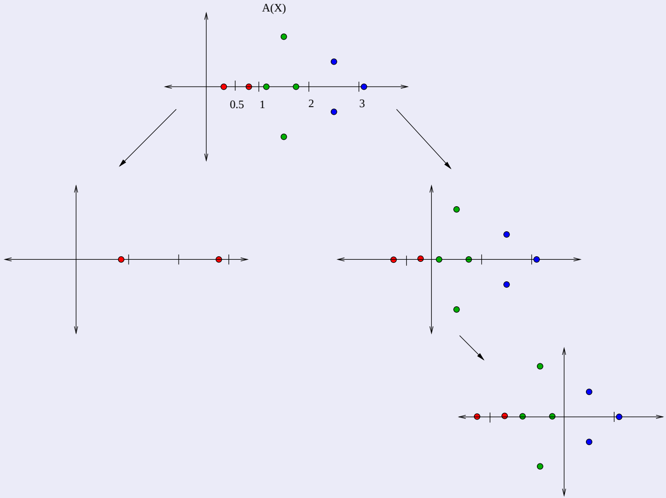
Construct $A_R(X) := A(X+1)$, $M_R(X) := X+1$. Check if $\texttt{Var}(A_R)$ is 0 or 1.



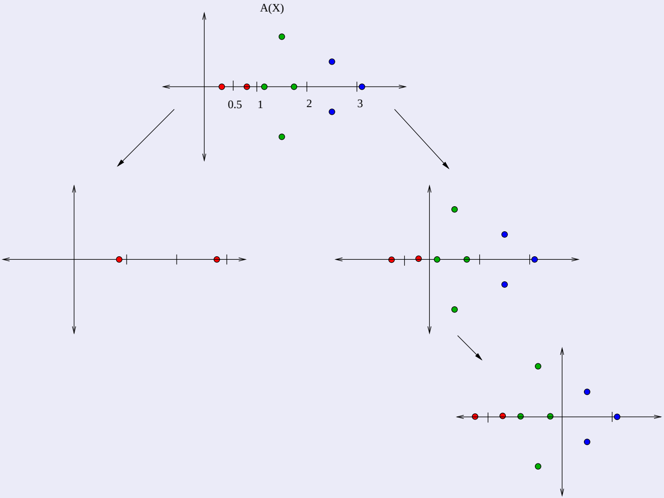
Construct $A_L(X) := (X+1)^n A\left(\frac{1}{X+1}\right)$, $M_L(X) := (X+1)^{-1}$.



Check if $\mathtt{Var}(A_L)$ is 0 or 1.



Construct $A_{RR}(X) := A_R(X+1) = A(X+2)$, $M_{RR}(X) := X+2$.



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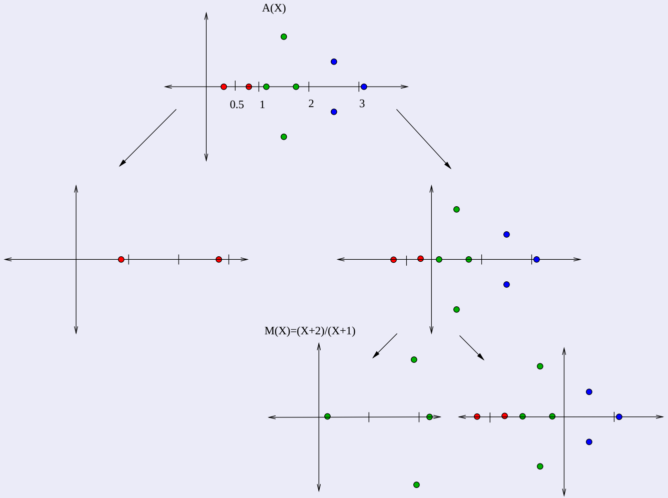
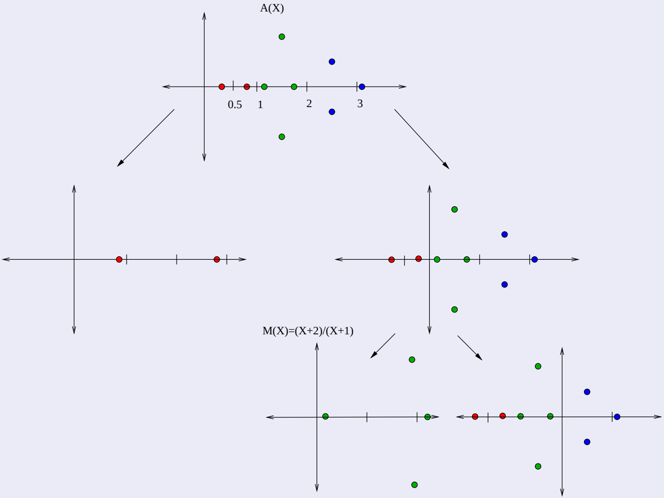
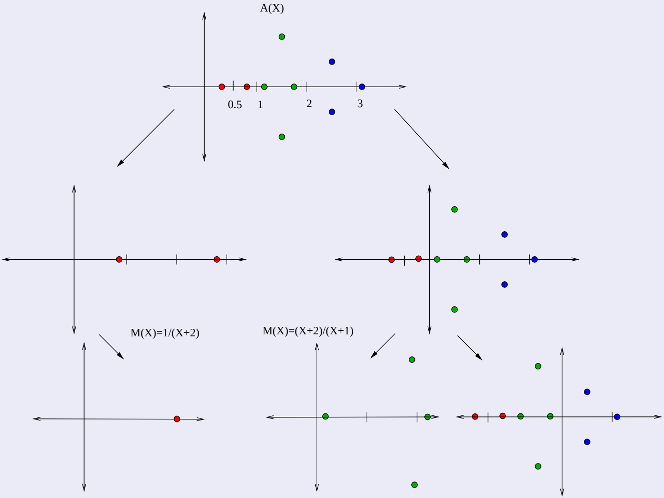


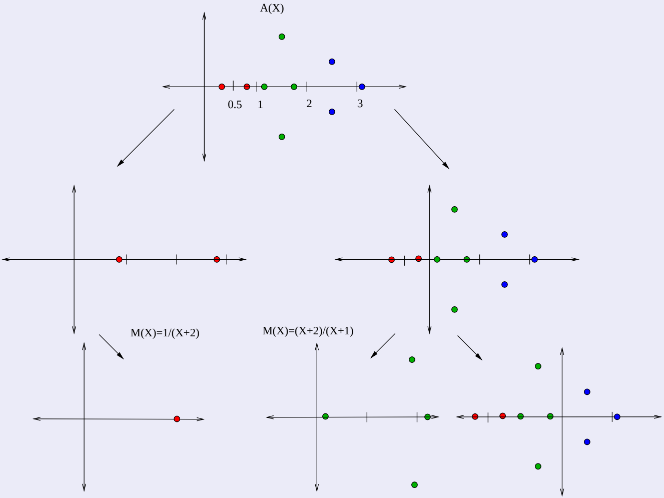
Construct $A_{RL}(X) := A_R(\frac{1}{1+X}) = A(1 + \frac{1}{1+X})$ and $M_{RL}(X) := 1 + \frac{1}{X+1}$.



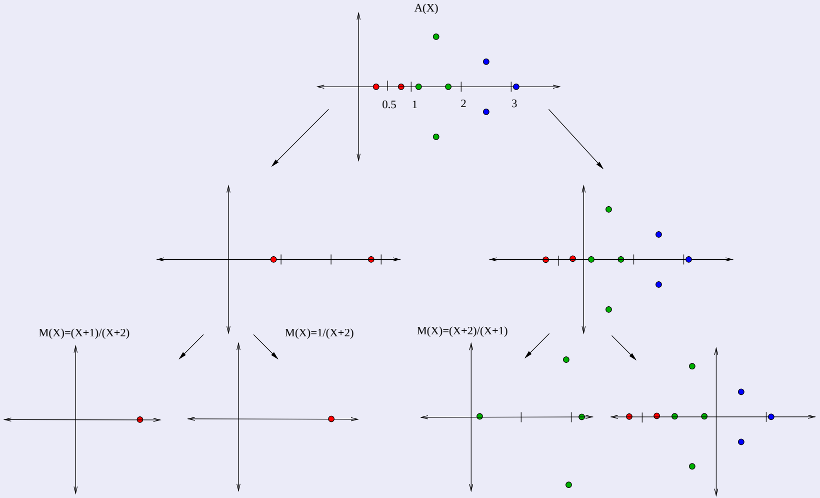
Check $\texttt{Var}(A_{RL})$.



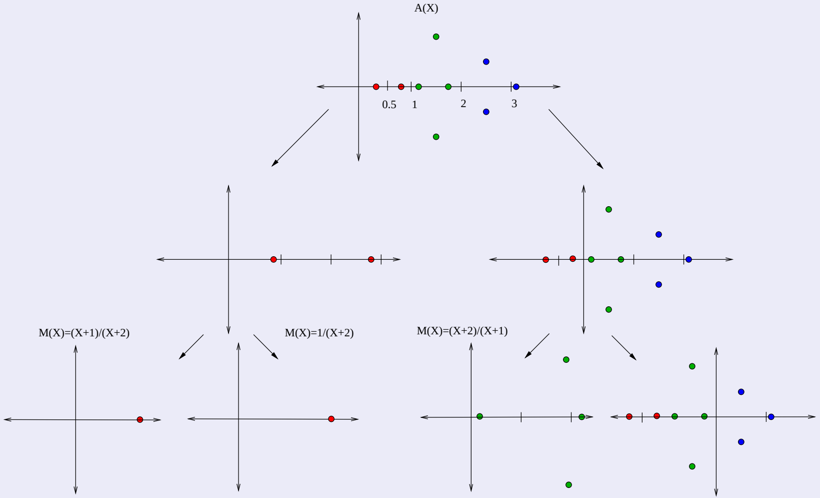
$$A_{LR}(X) = A_L(X+1) = (X+2)^n A\left(1 + \frac{1}{2+X}\right), M_{LR}(X) := (X+2)^{-1}$$



$\texttt{Var}(A_{LR}) = 1$, return $M_{LR}(0) = \frac{1}{2}$, $M_{LR}(\infty) = 0$



$$A_{RL}(X) := (X+1)^n A_L\left(\frac{1}{1+X}\right) = (X+2)^n A\left(\frac{1}{1+\frac{1}{1+X}}\right),\ M_{LR}(X) := \frac{X+1}{X+2}$$



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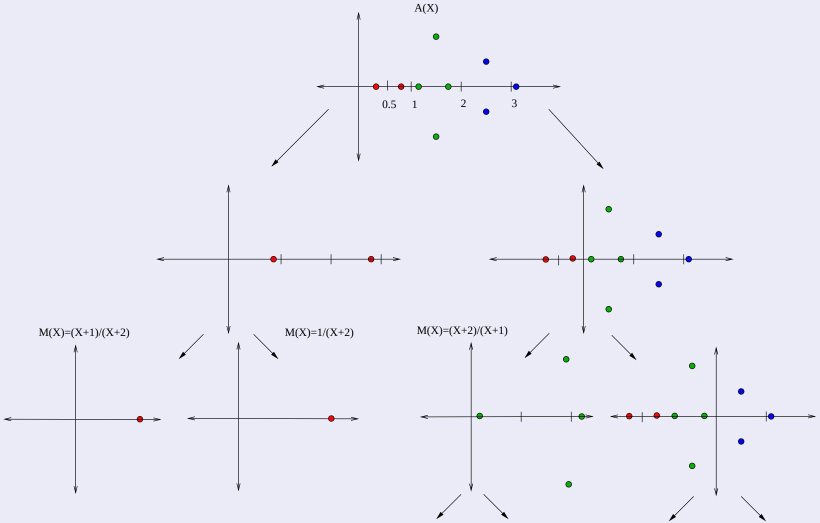
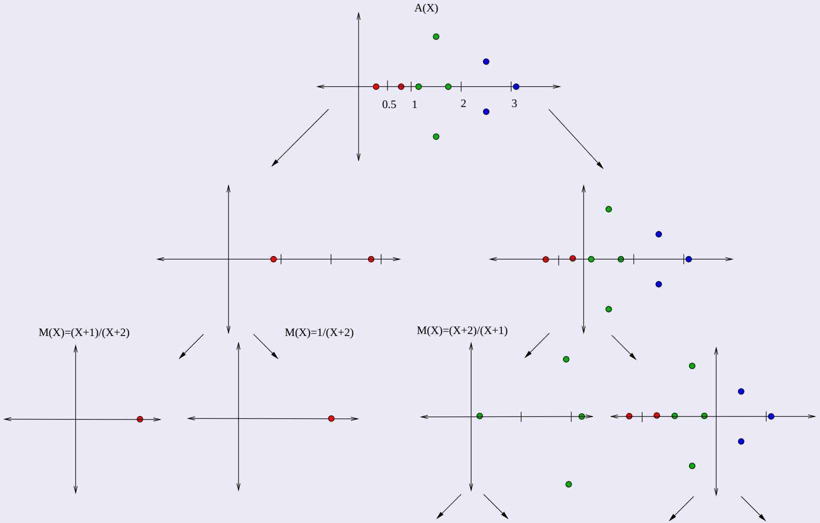
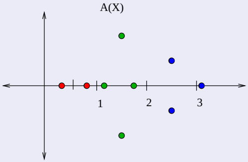


Continue recursively at each level



This was Uspensky's algorithm [Uspensky, 1948].

Vincent's Algorithm for Isolating Positive Roots



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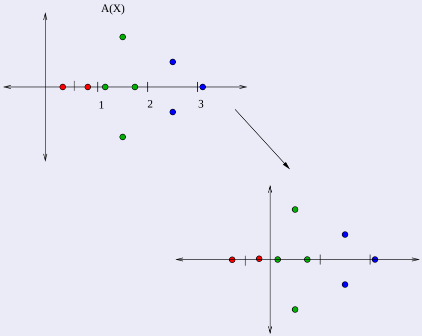

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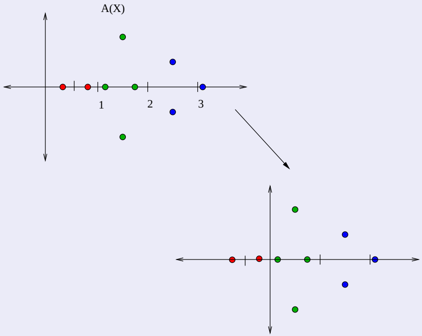

Is $\texttt{Var}(A_R) < \texttt{Var}(A)$?

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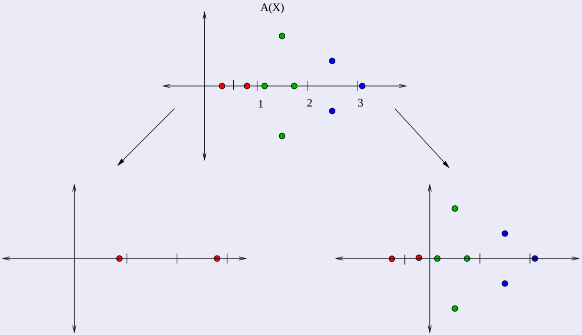

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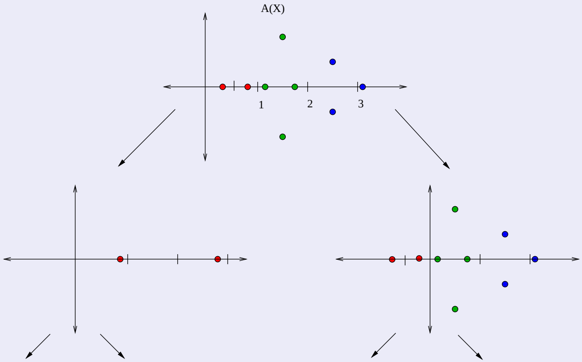

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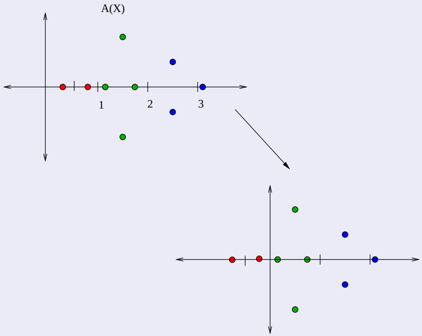

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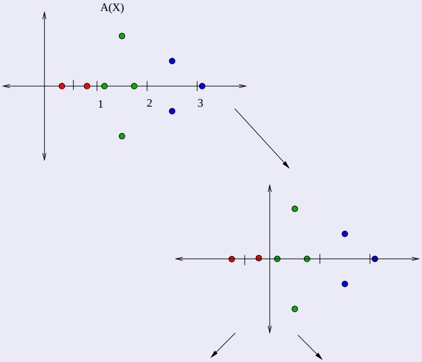

But proceed recursively from $A_R(X)$.

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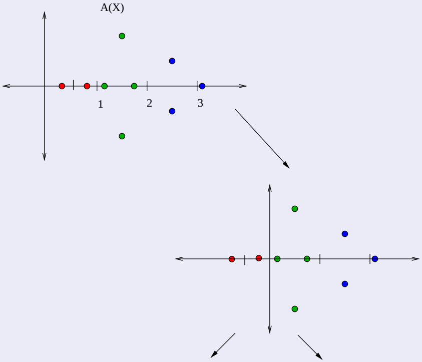

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Budan-Fourier

#(roots in $(0,1)$) $\leq \mathrm{Var}(A(X)) - \mathrm{Var}(A(X+1)) = \mathrm{Var}(A) - \mathrm{Var}(A_R)$.

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Advantages of Akritas' approach

- Faster in practice.
- Utilises distribution of roots.
- Computes the continued fraction approximation of the roots.

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- $RootIsol(A_R, M_R)$ and $RootIsol(A_L, M_L)$.

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Two steps for getting the worst-case bounds

1. Bound the worst-case size of the recursion tree:
   - number of inversion transformations, $X \rightarrow (X+1)^{-1}$ and
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2. Bound the worst-case complexity of a node in the recursion tree.

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Akritas' worst case bit-complexity

For $A(X) \in \mathbb{Z}[X]$, degree n, coefficients of bit-length L– $\widetilde{O}(n^4L^2)$:

- number of inversion transformations and Taylor shifts – $\widetilde{O}(n^2L)$.
- worst case bit-complexity of a node using fast integer arithmetic – $\widetilde{O}(n^2L)$.

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Drawbacks

- Assumes floor of the smallest positive root can be computed in $O(1)$.
- Assumes Taylor shifts don't increase the bit-size.

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Our worst case bit-complexity

Worst case bit-complexity is $\widetilde{O}(n^7 L^2)$:

- number of inversion transformations $\widetilde{O}(nL)$; no. of Taylor shifts $\widetilde{O}(n^3 L)$.
- worst case bit-complexity of a node using fast integer arithmetic

Akritas' Algorithm

$RootIsol(A,M)$

- If $\texttt{Var}(A) = 0$ **return**.
- If $\texttt{Var}(A) = 1$ **output** the interval with end points $M(0)$, $M(\infty)$.
- Compute a lower bound B on the positive roots of $A(X)$.
- If $B \geq 1$ then $A(X) := A(X+B)$, $M(X) := M(X+B)$.
- Compute $A_R(X) := A(X+1)$ and $M_R(X) := M(X+1)$.
- If $\mathrm{Var}(A_R) < \mathrm{Var}(A)$ then $A_L(X) := (X+1)^n A(\frac{1}{X+1})$, $M_L(X) := M(\frac{1}{X+1})$.
- $RootIsol(A_R, M_R)$ and $RootIsol(A_L, M_L)$.

- What are the transformations M_R, M_L?
- What is the relation between A_R, A_L and the input polynomial?

The transformation associated with a node in the tree

The transformation associated with a node in the tree

- Transformation associated with root is X.

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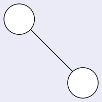

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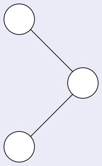
- Transformation associated with root is X.
- Do a Taylor shift $X \to X + a$.
- Then the transformation $X \to \frac{1}{1+X}$.



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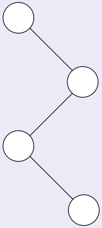

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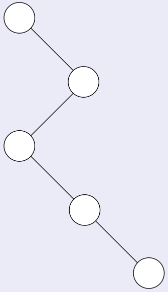

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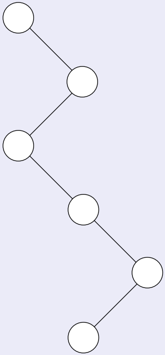

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- Associated transformation is
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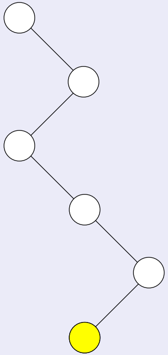

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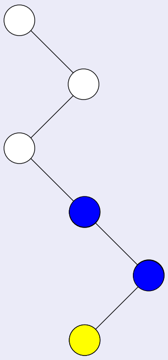

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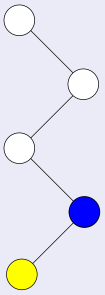
- Collapse consecutive Taylor shifts into one.



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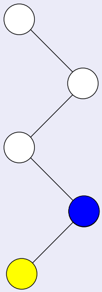

What is the transformation in general?

The transformation associated with a node in the tree

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_m + X}}}}$$

where

- m is the number of inversion transformations ($X \to \frac{1}{1+X}$).
- $q_0 \geq 0$ the total amount of Taylor shifts to the first inversion transformation.
- $q_i \geq 1$, for $i = 1, \ldots, m-1$, the total amount of Taylor shifts between i-th and $i+1$-th inversion transformation; if there are no Taylor shifts $q_i = 1$.

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Let ith quotient $\frac{P_i}{Q_i}$ be the finite continued fraction $q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \frac{1}{q_i}}}$.

Then $P_i = q_i P_{i-1} + P_{i-2}$ and $Q_i = q_i Q_{i-1} + Q_{i-2}$.

The transformation associated with a node in the tree

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_m + X}}}} = \frac{P_m X + P_{m-1}}{Q_m X + Q_{m-1}}.$$

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Two additional features associated with a node in the tree

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Let m be the number of inversion transformations along the path and

$$M(X) := \frac{P_m X + P_{m-1}}{Q_m X + Q_{m-1}}.$$

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- Polynomial $A_m(X) := (Q_m X + Q_{m-1})^n A(M(X))$.

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- Interval I_m that has end-points $M(0) = \frac{P_{m-1}}{Q_{m-1}}$, $M(\infty) = \frac{P_m}{Q_m}$.

Note: Width of I_m is $\left|\frac{P_m}{Q_m} - \frac{P_{m-1}}{Q_{m-1}}\right| = (Q_m Q_{m-1})^{-1}$.

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The positive roots of $A_m(X) \Leftrightarrow$ Roots of $A(X)$ in I_m.

$\mathtt{Var}(A_m) =$ #(number of roots of $A(X)$ in I_m) + even number.

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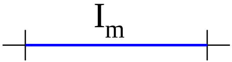
When does the algorithm terminate? When is $\mathrm{Var}(A_m) \leq 1$?

Termination Criterion: Two-Circle Theorem



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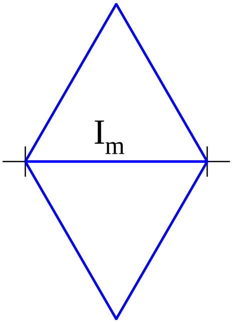

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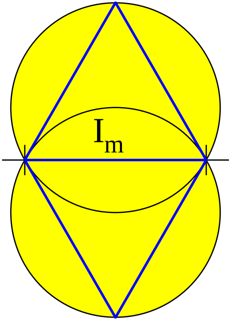

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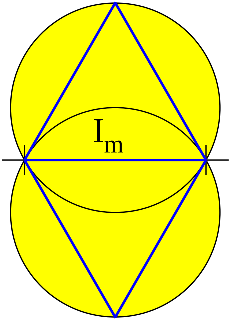

Two-circle Theorem ([Ostrowski, 1950])

If the two-circles figure w.r.t. I_m *contains a single root of* $A(X)$ *then* $\mathrm{Var}(A_m) = 1$*;*
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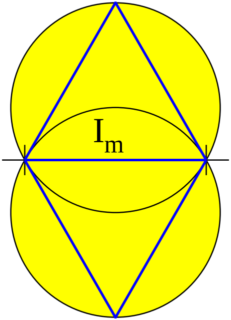

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If $\mathrm{Var}(A_m) \geq 2$*, then the two-circles figure in* $\mathbb{C}$ *w.r.t. interval* I_m *contains two roots* α, β *of* $A(X)$*.*

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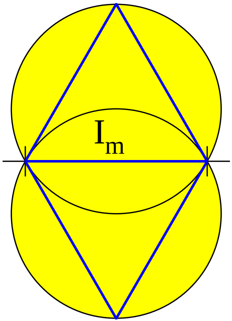

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We can choose a pair α, β of roots inside the two-circles such that

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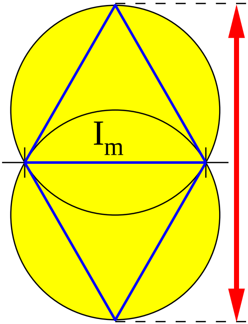

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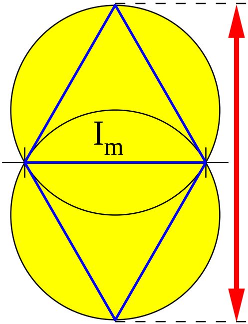

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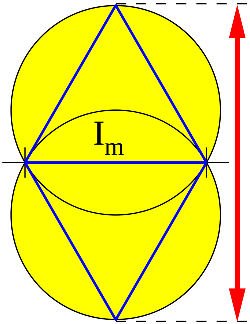

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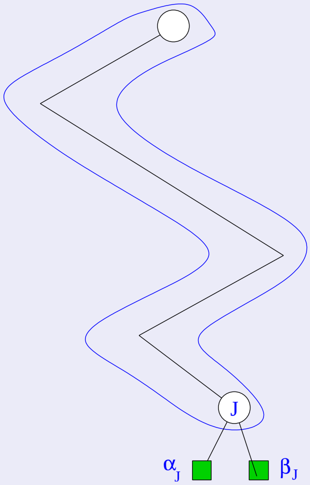
If $\mathrm{Var}(A_m) \geq 2$ then $\frac{1}{Q_m Q_{m-1}} > |\beta - \alpha|/\sqrt{3}$.

Number of Inversion Transformations along a path to a leaf



1. A path in the recursion tree of $RootIsol(A, X)$, from the root to a parent J of two leaves. Let m be the number of inversion transformations along the path.

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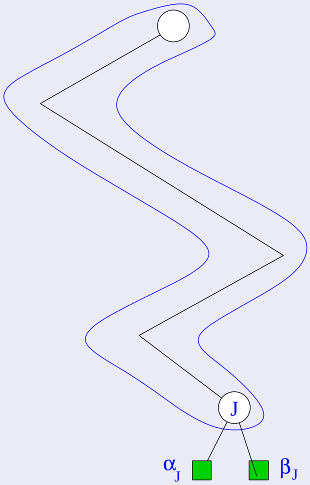

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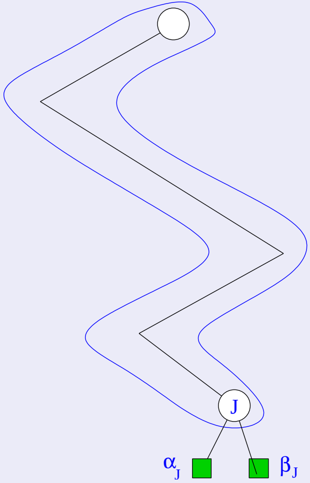

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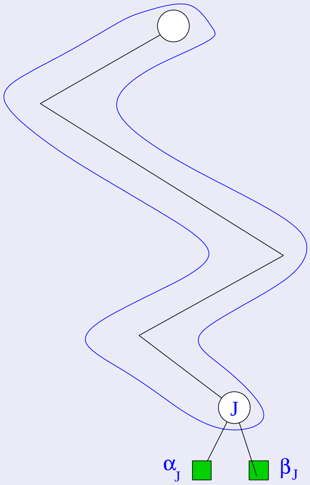

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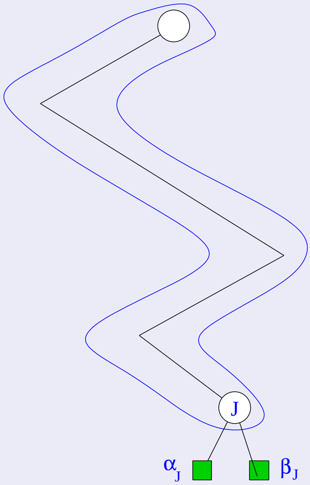

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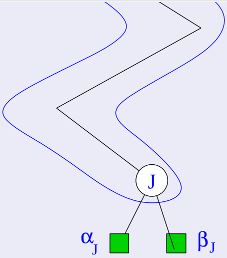
4. But $Q_m = q_m Q_{m-1} + Q_{m-2} \geq Q_{m-1} + Q_{m-2} \geq F_m \geq \phi^{m-1}$.
5. Thus $m \leq 2 - \log_\phi |\beta_J - \alpha_J|$.
6. This was shown by Uspensky and Ostrowski.

Number of Inversion Transformations along a path to a leaf



1. A path in the recursion tree of ... ent J ... th. ... J. ... oots (α_J, β_J) of $A(X)$ such that $|I_m| = \frac{1}{Q_m Q_{m-1}} \geq |\beta_J - \alpha_J|/\sqrt{3}$.
4. But $Q_m = q_m Q_{m-1} + Q_{m-2} \geq Q_{m-1} + Q_{m-2} \geq F_m \geq \phi^{m-1}$.
5. Thus $m \leq 2 - \log_\phi |\beta_J - \alpha_J|$.
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Proposition

The total number of inversion transformations in the tree are bounded by

$$\sum_J (2 - \log_\phi |\beta_J - \alpha_J|).$$

Akritas' Algorithm

$RootIsol(A,M)$

- If $\texttt{Var}(A) = 0$ **return**.
- If $\texttt{Var}(A) = 1$ **output** the interval with end points $M(0)$, $M(\infty)$.
- Compute a lower bound B on the positive roots of $A(X)$.
- If $B \geq 1$ then $A(X) := A(X+B)$, $M(X) := M(X+B)$.
- Compute $A_R(X) := A(X+1)$ and $M_R(X) := M(X+1)$.
- If $\mathrm{Var}(A_R) < \mathrm{Var}(A)$ then $A_L(X) := (X+1)^n A(\frac{1}{X+1})$, $M_L(X) := M(\frac{1}{X+1})$.
- $RootIsol(A_R, M_R)$ and $RootIsol(A_L, M_L)$.

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How do we compute a lower bound on positive roots of a polynomial?

Lower Bound on the smallest positive root

One Approach

- Roots of $X^n A(1/X)$ are inverse of the roots of $A(X)$.
- Compute an upper bound U on the largest positive root of $X^n A(1/X)$.
- $1/U$ is a lower bound on the smallest positive root of $A(X)$.

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Upper bound on positive roots, [Hong,98]

$B(X) = \sum_{i=0}^{n} b_i X^i$, $b_n > 0$. $U(B) := 2 \max_{b_i<0} \min_{b_j>0, j>i} \left\{ \left| \frac{b_i}{b_j} \right|^{1/(j-i)} \right\}$.

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Tight lower bound

$$\text{Define } \mathrm{PLB}(A) := \frac{1}{U(X^n A(1/X))}.$$

Suppose $A(X)$ has only real roots in $\Re(z) > 0$ and α is the smallest positive root of $A(X)$. Then

$$\frac{\alpha}{2n} \leq \mathrm{PLB}(A) < \alpha.$$

Bound on the number of Taylor shifts along a path

Assume the polynomial $A(X)$ has *only real roots*.

#(shifts needed to reach the floor of the smallest positive root α_0)?

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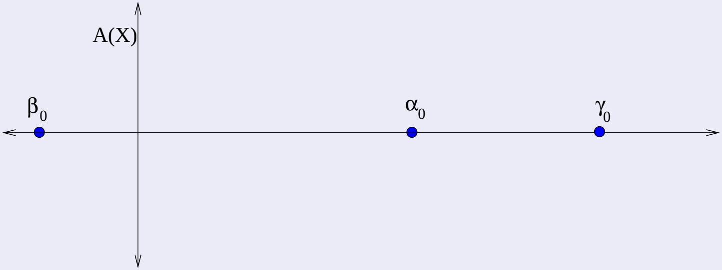

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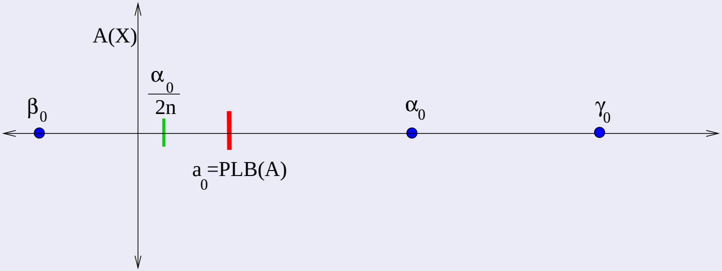

Since $\mathrm{PLB}(A)$ is a lower bound on α_0 we have $\frac{|\alpha_0|}{2n} \leq \mathrm{PLB}(A)$.

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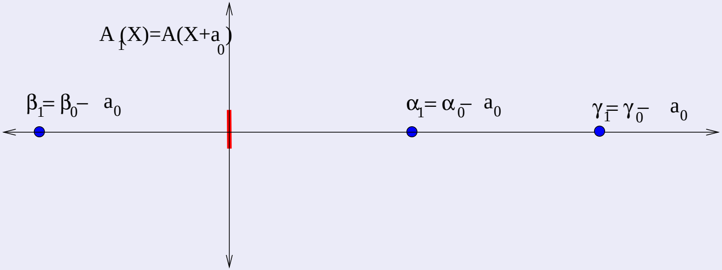

Shift $A(X)$ by $\mathrm{PLB}(A)$ to obtain $A_1(X)$.

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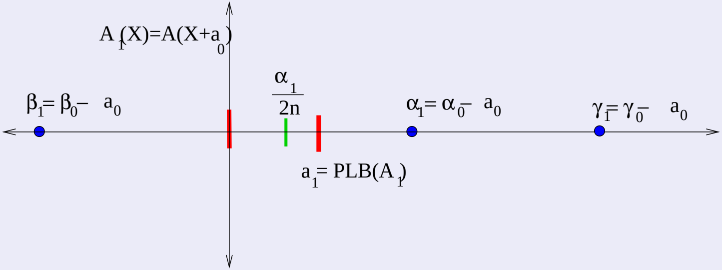

Suppose $\alpha_1 > 1$. Then $\frac{\alpha_1}{2n} \leq \mathrm{PLB}(A_1)$.

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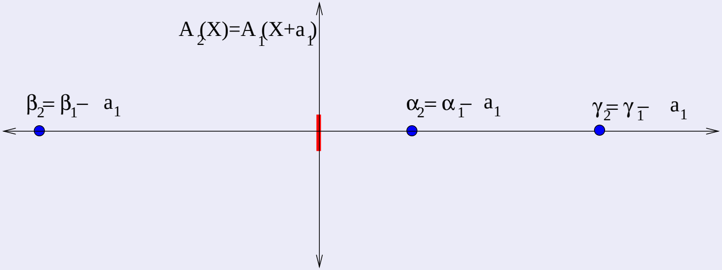

Shift $A_1(X)$ by a_1 to get $A_2(X)$.

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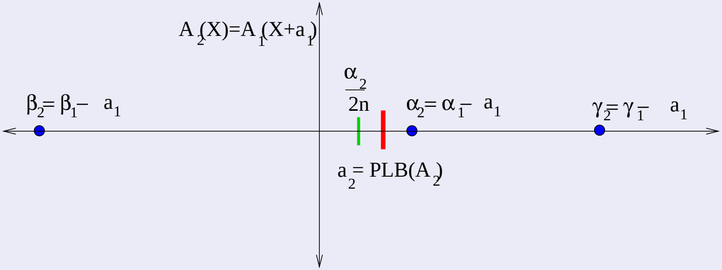
#(shifts needed to reach the floor of the smallest positive root α_0)?



Again suppose $\alpha_2 > 1$. Then $\frac{\alpha_2}{2n} \leq \mathrm{PLB}(A_2)$.

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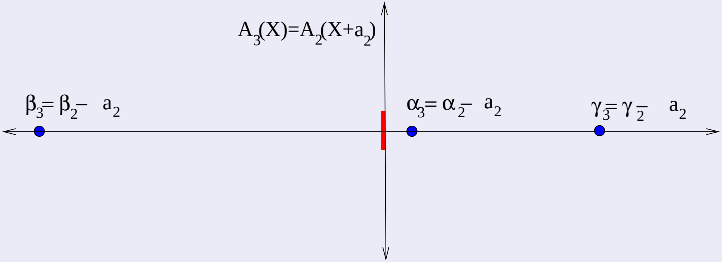

This continues until $\alpha_i < 1$, i.e., we compute the floor of α_0.

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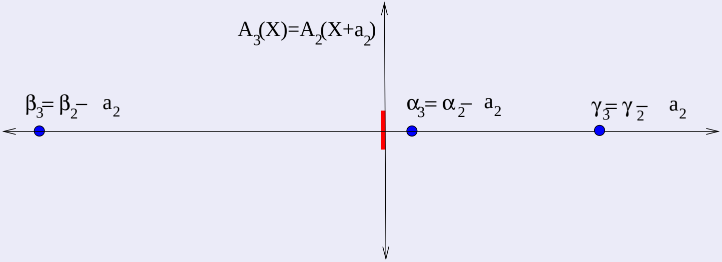

This continues until $\alpha_i < 1$, i.e., we compute the floor of α_0.

- $\alpha_i = \alpha_{i-1} - \mathrm{PLB}(A_{i-1}) \leq \alpha_{i-1}(1 - \frac{1}{2n})$.

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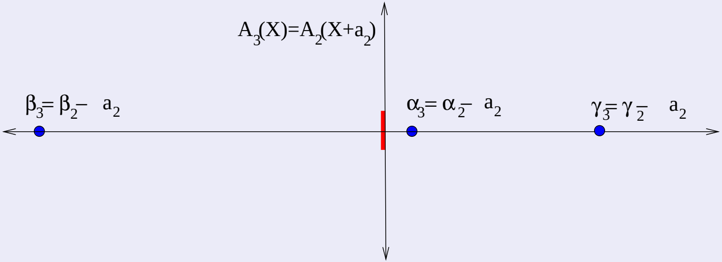

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- $\alpha_i = \alpha_{i-1} - \mathrm{PLB}(A_{i-1}) \le \alpha_{i-1}(1 - \frac{1}{2n})$.
- Thus $\alpha_i \le \alpha_0(1 - \frac{1}{2n})^i$.

Bound on the number of Taylor shifts along a path

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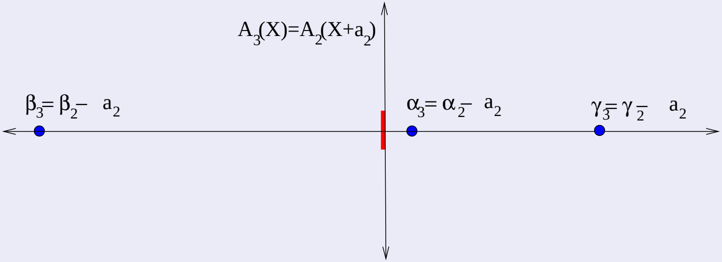
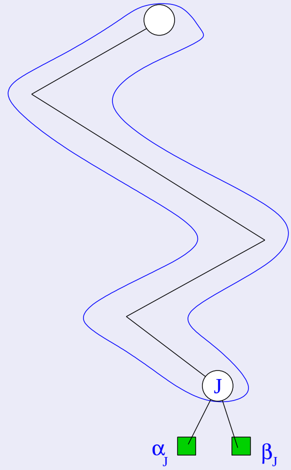


This continues until $\alpha_i < 1$, i.e., we compute the floor of α_0.

- $\alpha_i = \alpha_{i-1} - \mathrm{PLB}(A_{i-1}) \leq \alpha_{i-1}(1 - \frac{1}{2n})$.
- Thus $\alpha_i \leq \alpha_0(1 - \frac{1}{2n})^i$.
- Need at most $2n \log \alpha_0$ Taylor shifts to compute floor of α_0.

Number of Taylor shifts along a path in the tree

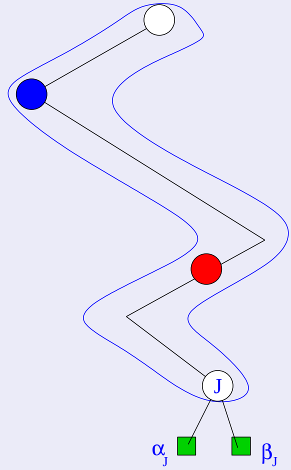
Assume the polynomial $A(X)$ has *only real roots*.



1. Consider a path in the recursion tree of $RootIsol(A, M(X))$, $M(X) = X$, from the root to a parent J of two leaves.
2. Let α_J, β_J be the roots associated with the leaves.
3. m be the number of inversion transformations along the path.

Number of Taylor shifts along a path in the tree

Assume the polynomial $A(X)$ has *only real roots*.



Consider the i-th and $i+1$-th inversion transformation.

$A_i(X)$ be the polynomial associated with the blue node. $a_1, \dots, a_\ell$ be its positive real roots.

#(Taylor shift) from i-th to $i+1$-th transformation is bounded by $2n(\log a_1 + \cdots + \log a_\ell) \leq 2n^2 \log a_\ell \leq 2n^2 \log q_i$.

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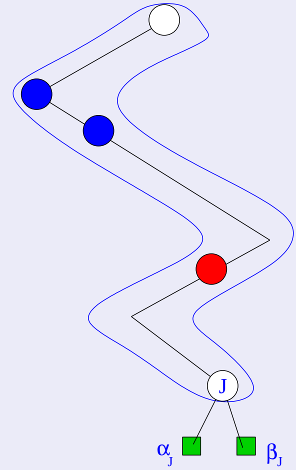

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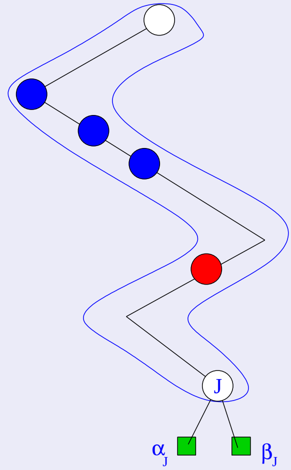

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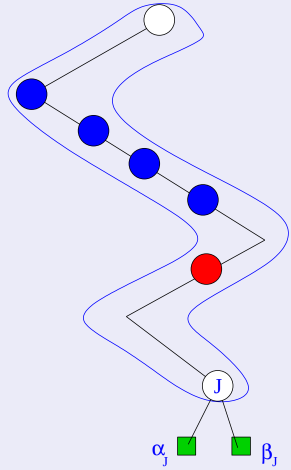

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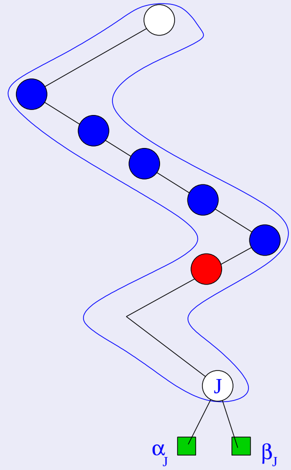

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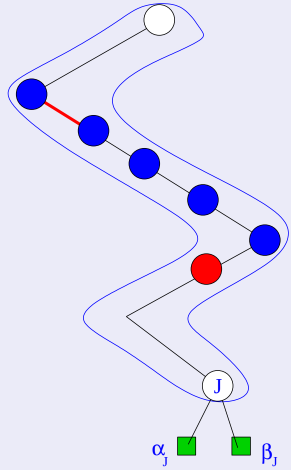

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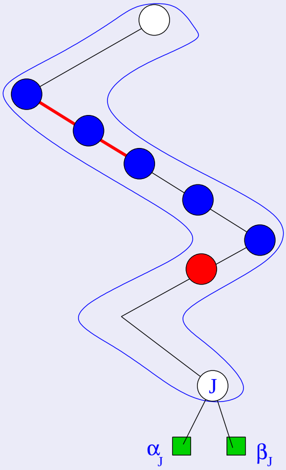

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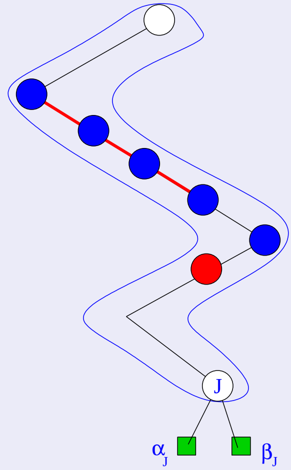

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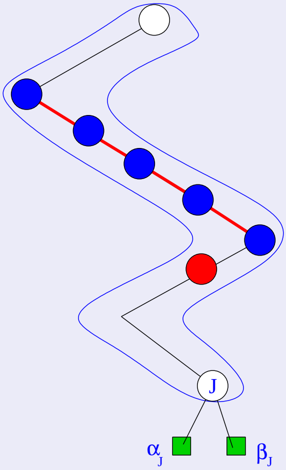

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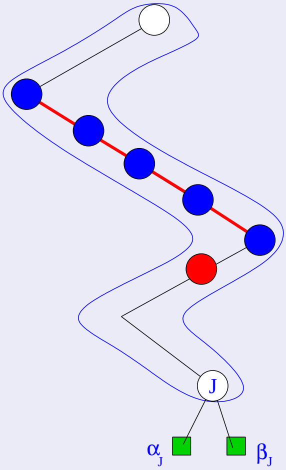

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Total number of Taylor shifts on the path to J is $n^2 O(\sum_{i=1}^{m} \log q_i)$.

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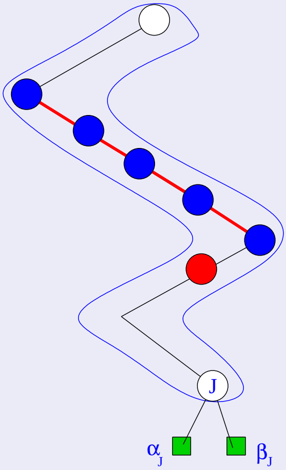

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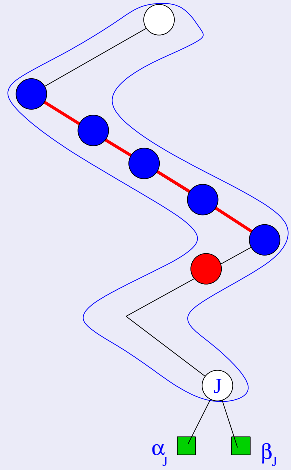

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Number of Taylor shifts along a path in the tree

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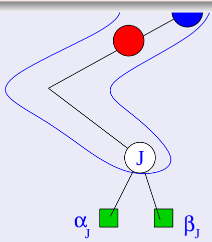
Proposition

The total number of Taylor shifts in the tree is bounded by

$$n^2 O(\sum_J \log|\alpha_J - \beta_J|^{-1}).$$

Lower bound on $\prod_J |\alpha_J - \beta_J|$*?*



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The Davenport–Mahler bound

Theorem (Davenport–Mahler [Dav., 1985] [Du/Sharma/Yap, 2005])

Consider a polynomial $A(X) \in \mathbb{C}[X]$ of degree n.
Let $G = (V, E)$ be a DAG whose vertices are the roots of $A(X)$. If

(i) $(\alpha, \beta) \in E \implies |\alpha| \leq |\beta|$, *and*

(ii) *in-degree of all vertices is at most one.*

then

$$\prod_{(\alpha,\beta)\in E} |\beta - \alpha| \geq \frac{\sqrt{|\mathrm{discr}(A)|}}{\mathrm{M}(A)^{n-1}} \cdot 2^{-O(n \log n)},$$

where, if ϑ_i are roots of $A(X)$,

$$\mathrm{discr}(A) := a_n^{2n-2} \prod_{i>j} (\vartheta_i - \vartheta_j)^2 \quad \textit{and} \quad \mathrm{M}(A) := |a_n| \prod_i \max\{1, |\vartheta_i|\}.$$

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Corollary

If $A(X) \in \mathbb{Z}[X]$ is square-free, has degree n, and coefficient bit-length L then
$\prod_{(\alpha,\beta)\in E} |\beta - \alpha| = 2^{-O(nL)}$.

Bound on the Size of the Recursion Tree

Assume $A(X)$ has degree n, coefficient bit-length L and only real roots.

Bound on the Size of the Recursion Tree

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Number of inversion transformations

$$\sum_J (2 - \log|\beta_J - \alpha_J|) = \widetilde{O}(nL).$$

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Let $A(X)$ be a square-free polynomial of degree n, and integer coefficients of bit-length L. The size of the recursion tree of Akritas' algorithm run on $A(X)$ is bounded by

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Bound on the Size of the Recursion Tree

Assume $A(X)$ has degree n, coefficient bit-length L and only real roots.

Number of inversion transformations

$$\sum_J (2 - \log|\beta_J - \alpha_J|) = \widetilde{O}(nL).$$

Number of Taylor shifts

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- The result holds in general!

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Combined with our worst-case bound $\widetilde{O}(n^3 L)$ on tree-size.

Main Result

Theorem

For a square-free integer polynomial of degree n, and coefficients of bit-length L, the worst-case running time of Akritas' algorithm is bounded by $\widetilde{O}(n^7L^2)$.

Expected Complexity, [Tsigaridas/Emiris, 2006]

The size of the tree

- Assumes floor of the smallest positive root can be computed in $O(1)$.
- Number of Taylor shifts $\sim$ Number of inversion transformations.
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Theorem

Expected running time of Akritas' algorithm:

- $\widetilde{O}(n^5 L^2)$ using classical Taylor shifts with fast integer arithmetic,
- $\widetilde{O}(n^4 L^2)$ using asymptotically fast Taylor shifts.

Comparison with the Descartes method

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Let $A(X) = \sum_{i=0}^{n} a_i \binom{n}{i} (X-c)^i (d-X)^{n-i} (d-c)^{-n}$.

$$E(A,(c,d)) := \#(\text{sign variations in } (a_n, a_{n-1}, \ldots, a_0)).$$

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Reasons

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- Lower bound is off by a factor of n (▸).

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But...

- Degree 100 Mignotte's polynomials ($X^n - (aX-1)^2$):
  [Emiris/Tsigaridas, '06]: Descartes 7.83sec. and Akritas 0.02sec.
- Available in `Mathematica`.
- [Collins/Akritas, 1976]: $O(n^6L^2)$; [Johnson, 1998]: $O(n^4L^2)$.

Possible ways to improve the complexity

1. Derive tight bounds on largest positive root of a polynomial in $O(n)$ operations. Bounds by [Kioustelidis, 1986; Ştefănescu, 2005] are known to be not tight. A recent bound by Akritas et al. might help.

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Open question

For Mignotte's polynomial $X^n - 2(aX-1)^2$, $a \in \mathbb{N}$, the size of the recursion tree is $O(\log a)$ using Zassenhaus' bound
(the Descartes method has recursion tree size $\Omega(n \log a)$).

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Main Result

For a square-free polynomial $A(X)$, degree n, and coefficient bit-length L:

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Paper is available from `http://www.cs.nyu.edu/sharma/pap/`.

Merci!