Bounds on List Decoding of Rank Metric Codes

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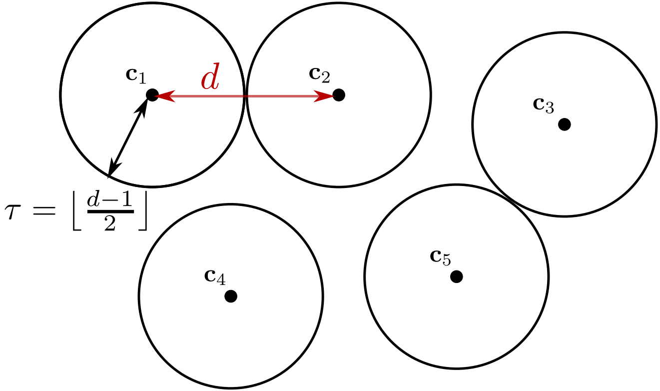
Motivation — Reed–Solomon vs. Gabidulin Codes

For a code $\mathcal{C}$ of length n, dimension k and minimum distance d, unique decoding is possible up to $\tau = \left\lfloor \frac{d-1}{2} \right\rfloor$.



What about decoding algorithms for Gabidulin codes?
Similar to Reed–Solomon codes?

Reed–Solomon vs. Gabidulin Codes — Algorithms

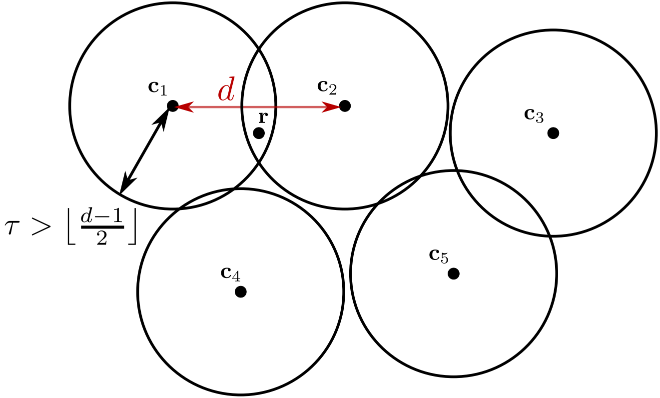
Decoding up to half the minimum distance $\tau = \left\lfloor \frac{d-1}{2} \right\rfloor$

	Reed–Solomon Codes	Gabidulin Codes
System of equations	Peterson, . . .	Gabidulin
Shift–Register Synthesis	Berlekamp–Massey	Paramonov–Tretjakov, Richter–Plass
Euclidean Algorithm	Sugiyama, . . .	Gabidulin
Interpolation	Welch–Berlekamp	Loidreau
⋮	⋮	⋮

Many parallels between Reed–Solomon and Gabidulin codes!

List Decoding

For a code $\mathcal{C}$ of length n, dimension k and minimum distance d, there can be several codewords in a ball of radius $\tau > \left\lfloor \frac{d-1}{2} \right\rfloor$.



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	Reed–Solomon Codes	Gabidulin Codes
Interpolation (List Decoding)	Sudan Guruswami–Sudan (and many accelerations)	?
Syndrome-based (Unique Decoding)	Schmidt–Sidorenko	

Is polynomial–time list decoding possible for Gabidulin codes?

Outline

Outline

Rank Metric

Rank Metric

- Let $\mathcal{B}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ where q is a power of a prime
- Each vector $\mathbf{x} \in \mathbb{F}_{q^m}^n$ can be mapped on a matrix $\mathbf{X} \in \mathbb{F}_q^{m \times n}$
- Rank norm: $\mathrm{rank}(\mathbf{x}) \overset{\mathrm{def}}{=}$ rank of $\mathbf{X}$ over $\mathbb{F}_q$

Minimum Rank Distance of a block code $\mathcal{C}$:

- $d \overset{\mathrm{def}}{=} \min\{\mathrm{rank}(\mathbf{c}_1 - \mathbf{c}_2) \mid \mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}, \mathbf{c}_1 \neq \mathbf{c}_2\} \leq n - k + 1$
- Codes with $d = n - k + 1$ are called Maximum Rank Distance (MRD) codes

Linearized Polynomial over $\mathbb{F}_{q^m}$

- $f(x) \overset{\mathrm{def}}{=} \sum_{i=0}^{d_f} f_i x^{[i]} = \sum_{i=0}^{d_f} f_i x^{q^i}$ with $f_i \in \mathbb{F}_{q^m}$.
- If $f_{d_f} \neq 0$, define the q-degree: $\deg_q f(x) = d_f$.

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Gabidulin Codes

Introduced by *Delsarte* (1978), *Gabidulin* (1985), *Roth* (1991)

- A linear Gabidulin code $\mathcal{G}(n,k)$ of length $n \leq m$ and dimension k over $\mathbb{F}_{q^m}$ is defined by

$$\mathcal{G}(n,k) \overset{\text{def}}{=} \left\{ \mathbf{c} = (f(\alpha_0)\ f(\alpha_1)\ \ldots\ f(\alpha_{n-1})) \,\middle|\, \deg_q f(x) < k \right\},$$

  where all $f(x)$ are linearized polynomials and $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{F}_{q^m}$ are linearly independent over $\mathbb{F}_q$.

Minimum Rank Distance of a Gabidulin Code

- $d = \min\{\mathrm{rank}(\mathbf{c}) \mid \mathbf{c} \in \mathcal{G}, \mathbf{c} \neq \mathbf{0}\} = n - k + 1.$

Outline

Problem Statement

Is polynomial–time list decoding possible for rank metric codes (and in particular for Gabidulin codes)?

Problem (Maximum List Size)

Let $\mathcal{C}(n, M, d)$ be a code over $\mathbb{F}_{q^m}$ with $n \leq m$ and minimum rank distance d. Let $\tau < d$. Find a lower and upper bound on the maximum number of codewords ℓ in a ball of rank radius τ. Hence, find a bound on

$$\ell \stackrel{\text{def}}{=} \max_{\mathbf{r} \in \mathbb{F}_{q^m}^n} \left(|\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{C}(n, M, d)| \right).$$

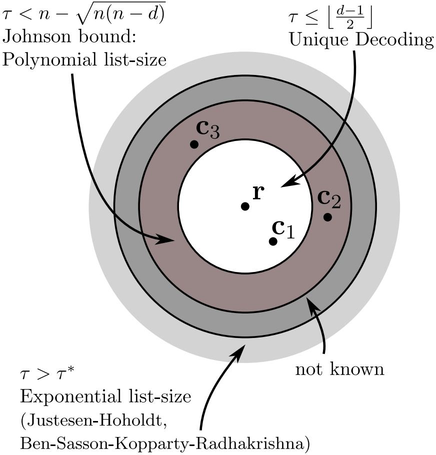
Interpretation:

- Lower exponential bound: no polynomial-time list decoding,
- Upper polynomial bound: polynomial-time list decoding might exist.

Outline

Bounds on the Maximal List-Size

Reed–Solomon codes



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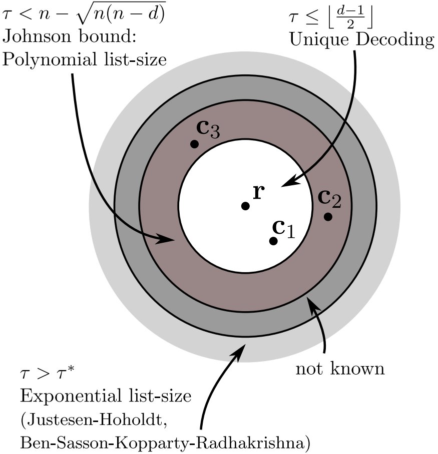

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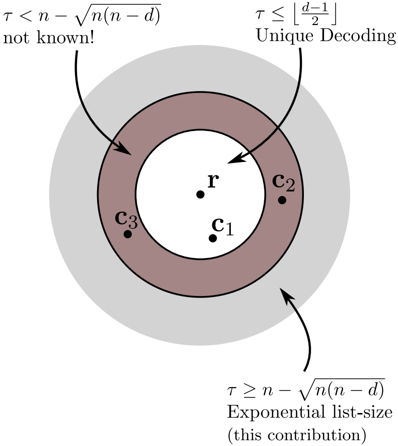

A Lower Bound on the List Size

Theorem (Lower Bound on the List Size)

Let the Gabidulin code $\mathcal{G}(n,k)$ over $\mathbb{F}_{q^m}$ with $n \leq m$ and $d = n-k+1$ be given and let $\tau < d$. Then, there exists a word $\mathbf{r} \in \mathbb{F}_{q^m}^n$ such that

$$\ell \geq |\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{G}(n,k)| \geq \frac{\begin{bmatrix} n \\ n-\tau \end{bmatrix}}{(q^m)^{n-\tau-k}} \geq q^m q^{\tau(m+n)-\tau^2-md},$$

and for the special case of $n = m$: $\ell \geq q^n q^{2n\tau-\tau^2-nd}$.

- For $n = m$ this is $\ell \geq q^{n(1-\epsilon)} \cdot q^{2n\tau-\tau^2-nd+n\epsilon}$
- Exponential in n if $\tau \geq n - \sqrt{n(n-d+\epsilon)}$ and $0 \leq \epsilon < 1$ (= Johnson radius).
- Proof similar to the proof of Justesen-Hoholdt for RS codes.

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A Lower Bound on the List Size – Proof

Proof (i)

- $\mathcal{P}^* \stackrel{\text{def}}{=}$ set of all monic linearized polynomials of $\deg_q = n - \tau$ and a root space over $\mathbb{F}_{q^n}$ of dimension $n - \tau > k - 1$
- $|\mathcal{P}^*| = \begin{bmatrix} n \\ n-\tau \end{bmatrix}$
- $\mathcal{P} \stackrel{\text{def}}{=}$ subset of $\mathcal{P}^*$ such that all q-monomials of q-degree greater than or equal to k have the same coefficients
- There are $(q^m)^{n-\tau-k}$ possibilities to choose the highest $n - \tau - (k-1)$ coefficients
- There exist coefficients such that $|\mathcal{P}| \geq \frac{\begin{bmatrix} n \\ n-\tau \end{bmatrix}}{(q^m)^{n-\tau-k}}$
- For any $f(x), g(x) \in \mathcal{P}$, $\deg_q(f(x) - g(x)) < k$, is evaluation polynomial of a codeword of $\mathcal{G}(n, k)$

. . .

A Lower Bound on the List Size – Proof

Proof (ii)

- Let $f(x), g(x) \in \mathcal{P}$
- Let $\mathcal{A} = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$
- Let $\mathbf{r} = (r_0 \; r_1 \; \dots \; r_{n-1}) = (f(\alpha_0) \; f(\alpha_1) \; \dots \; f(\alpha_{n-1}))$
- Let $\mathbf{c}$ be the evaluation of $f(x) - g(x)$ at $\mathcal{A}$
- Then, $\mathbf{r} - \mathbf{c}$ is the evaluation of $f(x) - f(x) + g(x) = g(x) \in \mathcal{P}$, whose root space has dimension $n - \tau$ and all roots are in $\mathbb{F}_{q^n}$
- $\dim \ker(\mathbf{r} - \mathbf{c}) = n - \tau$ and $\mathrm{rk}(\mathbf{r} - \mathbf{c}) = \tau$

Therefore, for **any** $g(x) \in \mathcal{P}$, the evaluation of $f(x) - g(x)$ is a codeword from $\mathcal{G}(n, k)$ and has rank distance τ from $\mathbf{r}$.

$$\Longrightarrow \boldsymbol{\ell} \geq |\boldsymbol{\mathcal{P}}| \geq \frac{\begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{n-\tau} \end{bmatrix}}{(\boldsymbol{q^m})^{\boldsymbol{n-\tau-k}}}. \qquad \square$$

Outline

An Upper Bound on the List Size

Theorem (Upper Bound on the List Size)

Let **any** *rank metric code* $\mathcal{C}(n, M, d)$ *over* $\mathbb{F}_{q^m}$ *with* $n \leq m$ *and minimum rank distance* d *be given. Let* $\tau < d$. *Then, for* **any** *word* $\mathbf{r} \in \mathbb{F}_{q^m}^n$ *and hence, for the maximum list size, the following holds*

$$\ell = \max_{\mathbf{r} \in \mathbb{F}_{q^m}^n} \left(|\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{C}(n, M, d)| \right) \leq \sum_{t=\left\lfloor \frac{d-1}{2} \right\rfloor + 1}^{\tau} \frac{\begin{bmatrix} n \\ 2t+1-d \end{bmatrix}}{\begin{bmatrix} t \\ 2t+1-d \end{bmatrix}}$$

$$\leq 4 \sum_{t=\left\lfloor \frac{d-1}{2} \right\rfloor + 1}^{\tau} q^{(2t-d+1)(n-t)}.$$

- Exponential in $n \leq m$ for any $\tau > \lfloor (d-1)/2 \rfloor$
- Does not provide any conclusion if polynomial-time list decoding is possible or not up to the Johnson bound.

A Lower Bound on the List Size

Theorem (Lower Bound on the List-Size)

Let $n \leq m$, $\tau \geq \lfloor (d-1)/2 \rfloor + 1$ *and* $\tau \leq n - \tau$.
Then, there exists a rank metric code $\mathcal{C}(n, M, d_R \geq d)$ *over* $\mathbb{F}_{q^m}$ *of length* n *and minimum rank distance* $d_R \geq d$, *and a word* $\mathbf{r} \in \mathbb{F}_{q^m}^n$ *such that*

$$\left| \mathcal{C}(n, M, d_R \geq d) \cap \mathcal{B}_\tau(\mathbf{r}) \right| \geq q^{(n-\tau)(\tau - \lfloor (d-1)/2 \rfloor)}.$$

- Shows there exists a rank metric code and a received word such that list size is **exponential** in n for $\tau > \lfloor (d-1)/2 \rfloor$.
  $\Rightarrow$ No polynomial-time list decoding for these codes!
- $\mathcal{C}(n, M, d_R \geq d)$ might be non-linear and non-MRD.
- The restriction $\tau \leq n - \tau$ is always fulfilled for $\tau = \lfloor (d-1)/2 \rfloor + 1$ and $k > 1$.
- Proof uses interpretation of $\{\mathbf{r} - \mathbf{c}_1, \mathbf{r} - \mathbf{c}_2, \dots, \mathbf{r} - \mathbf{c}_\ell\}$ as constant-rank code of rank τ and minimum rank distance d.

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Conclusion

We have shown three bounds on the list size of rank metric codes:

The lower bound for **Gabidulin codes**

- is based on the evaluation of linearized polynomials,
- shows that polynomial-time list decoding is not possible for $\tau \geq n - \sqrt{n(n-d+\epsilon)}$.

The upper bound for **any** rank metric code

- uses subspace properties,
- is exponential in n.

The lower bound for rank metric codes

- uses the interpretation as constant-rank code,
- **shows that there exists a rank metric code with exponential list size for $\tau \geq \lfloor (d-1)/2 \rfloor + 1$.**

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Merci pour votre attention !